# 108 ispitnih zadataka za vježbu podjeljenih po oblastima - detaljno raspisana rješenja ovih zadataka možete skinuti sa stranice ff.unze.ba\nabokov\za_vjezbu

## Sadržaj

# 1 Determinante

**1.** Izračunati determinantu $D = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 4 & 4 \\ -2 & -2 & -2 & 1 \\ 3 & 3 & 6 & x^2+3 \end{vmatrix}$, a zatim riješiti nejednačinu $D < 2x$.

**2.** Izračunati determinantu $D = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & 6 & 5 \\ -1 & -1 & 0 & 2 \\ -3 & -3 & -6 & x^2-3 \end{vmatrix}$, a zatim riješiti nejednačinu $D < 2x$.

**3.** Izračunati determinantu $D = \begin{vmatrix} x^2-8 & 2 & 3 & 1 \\ 4 & 2 & 3 & 1 \\ -4 & -2 & 1 & 0 \\ 5 & 3 & 2 & 1 \end{vmatrix}$, a zatim riješiti nejednačinu $D < -x$.

**4.** Izračunati determinantu $D = \begin{vmatrix} 2 & 1 & 1 & -1 \\ 4 & x^2-1 & 1 & -1 \\ -4 & -2 & -1 & 0 \\ 5 & 3 & 2 & -1 \end{vmatrix}$, a zatim riješiti nejednačinu $D > x$.

**5.** Izračunati determinantu $D = \begin{vmatrix} 1 & -2 & 2 & 1 \\ 3 & -7 & 6 & 2 \\ -4 & 6 & -9 & -6 \\ 5 & -7 & 12 & x^2 \end{vmatrix}$, a zatim riješiti nejednačinu $D > -2x$.

**6.** Izračunati determinantu $D = \begin{vmatrix} 1 & -2 & 2 & 1 \\ 3 & -7 & 6 & 2 \\ -4 & 6 & x^2-13 & -6 \\ 5 & -7 & 12 & 7 \end{vmatrix}$, a zatim riješiti nejednačinu $D < 4x$.

# 2 Matrične jednačine

**7.** Riješiti matričnu jednačinu $BX = A + I$ ako je $A = \begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 7 & -3 \\ 6 & 9 & -1 \end{bmatrix}$ i $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

**8.** Riješiti matričnu jednačinu $2I + BX = A$ ako je $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -1 \\ -2 & -3 & 1 \\ -3 & -6 & 4 \end{bmatrix}$ i $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

**9.** Riješiti matričnu jednačinu $-3X = 2AX + I$ ako je $A = \begin{bmatrix} -2 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ i $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

**10.** Riješiti matričnu jednačinu $AX+I=-3X$ ako je $A=\begin{bmatrix} 11 & -2 & -13 \\ -6 & -2 & 5 \\ -1 & 0 & -2 \end{bmatrix}$ i $I=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

**11.** Riješiti matričnu jednačinu $I+AX=-2X$ ako je $A=\begin{bmatrix} -8 & -2 & 9 \\ -3 & -3 & 4 \\ 1 & 0 & -3 \end{bmatrix}$ i $I=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

**12.** Riješiti matričnu jednačinu $-2X=3AX-I$ ako je $A=\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 4 & -1 & -1 \end{bmatrix}$ i $I=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

**13.** Riješiti matrične jednačine

$$(a)\ X\cdot(-2)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+X\begin{bmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \end{bmatrix}\begin{bmatrix} -1 & -8 \\ 7 & 1 \\ -2 & -4 \end{bmatrix};$$

$$(b)\ X\begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 8 \\ -7 & -1 \\ 2 & 4 \end{bmatrix}=X\cdot(-1)+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

$$(c)\ \begin{bmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -8 \\ 7 & -1 \\ 2 & -4 \end{bmatrix}X=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-2X;$$

$$(d)\ 3X=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} -1 & 8 \\ -7 & 1 \\ 2 & 4 \end{bmatrix}X+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

**14.** Riješiti matrične jednačine

$(a)$ $CXA+XB=A$ ako su $A=\begin{bmatrix} 0 & -1 & 2 \\ 2 & -1 & -3 \\ -1 & 1 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 0 & -3 & 6 \\ 6 & -3 & -9 \\ -3 & 3 & 3 \end{bmatrix}$ i $C=\begin{bmatrix} -1 & 3 & 5 \\ 1 & -1 & 4 \\ 1 & 1 & -1 \end{bmatrix}$;

$(b)$ $CXB+AX=C$ ako su $A=\begin{bmatrix} 0 & -2 & -4 \\ -4 & -2 & -6 \\ -2 & 0 & -2 \end{bmatrix}$, $B=\begin{bmatrix} -1 & -1 & 1 \\ 1 & -4 & 4 \\ -1 & 1 & -4 \end{bmatrix}$ i $C=\begin{bmatrix} 0 & -1 & -2 \\ -2 & -1 & -3 \\ -1 & 0 & -1 \end{bmatrix}$;

$(c)$ $AXC+XB=C$ ako su $A=\begin{bmatrix} 0 & 3 & 5 \\ 1 & 0 & 4 \\ 1 & 1 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ -1 & 1 & -2 \end{bmatrix}$ i $C=\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & 2 \\ -\frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}$.

**15.** Riješiti matrične jednačine

$$(a)\ A^{-1}XB = 2A^{-1}X + I \text{ ako su } A = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix} \text{ i } B = \begin{bmatrix} 6 & 7 & 4 \\ 2 & 6 & 2 \\ 2 & 4 & 5 \end{bmatrix}.$$

$$(b)\ AXB^{-1} = 2XB^{-1} - I \text{ ako su } A = \begin{bmatrix} 6 & 7 & 4 \\ 2 & 6 & 2 \\ 2 & 4 & 5 \end{bmatrix} \text{ i } B = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix}.$$

# 3 Sistemi linearnih jednačina

**16.** Rješiti sistem linearnih jednačina

$$\begin{aligned} x - y + z &= 3 \\ x - y - z &= 4 \\ x + y - z &= 5 \\ x + y + z &= 6\ . \end{aligned}$$

**17.** Rješiti sistem linearnih jednačina

$$\begin{aligned} x - y + z &= 2 \\ x - y - z &= 3 \\ x + y - z &= 4 \\ x + y + z &= 5\ . \end{aligned}$$

**18.** Rješiti sistem linearnih jednačina

$$\begin{aligned} x - y + z &= 1 \\ x - y - z &= 2 \\ x + y - z &= 3 \\ x + y + z &= 4\ . \end{aligned}$$

**19.** Riješiti sistem jednačina

$$\begin{aligned} x_1 + 2x_2 - x_3 - x_4 - x_5 &= 10 \\ x_1 + 3x_2 + x_3 + x_4 + x_5 &= 20 \\ -3x_1 - 6x_2 + 4x_3 + 4x_4 + 4x_5 &= -27\ . \end{aligned}$$

**20.** Riješiti sistem jednačina

$$\begin{aligned} x_1 - 2x_2 + x_3 + x_4 + x_5 &= 0 \\ -x_1 + 3x_2 - 3x_3 - 3x_4 - 3x_5 &= -2 \\ 3x_1 - 6x_2 + 4x_3 + 4x_4 + 4x_5 &= 3\ . \end{aligned}$$

**21.** Riješiti sistem jednačina

$$\begin{aligned} x_1 + 2x_2 - 4x_3 - 8x_4 - 12x_5 &= -11 \\ -2x_1 - 3x_2 + 5x_3 + 10x_4 + 15x_5 &= 7 \\ -3x_1 - 5x_2 + 10x_3 + 20x_4 + 30x_5 &= 25. \end{aligned}$$

**22.** Riješiti sistem jednačina

$$\begin{aligned} x_1 + 2x_2 - 4x_3 + 8x_4 + 12x_5 &= -10\\ 3x_1 + 7x_2 - 15x_3 + 30x_4 + 45x_5 &= -43\\ -2x_1 - 3x_2 + 6x_3 - 12x_4 - 18x_5 &= 13. \end{aligned}$$

**23.** Riješiti sistem jednačina

$$\begin{aligned} x_1 + 2x_2 - 4x_3 - 4x_4 - 16x_5 &= -9\\ 2x_1 + 5x_2 - 11x_3 - 11x_4 - 44x_5 &= -29\\ -4x_1 - 7x_2 + 14x_3 + 14x_4 + 56x_5 &= 30. \end{aligned}$$

**24.** Riješiti sistem jednačina

$$\begin{aligned} x_1 + 2x_2 - 4x_3 + 16x_4 - 4x_5 &= -8\\ -2x_1 - 3x_2 + 5x_3 - 20x_4 + 5x_5 &= 7\\ 4x_1 + 9x_2 - 18x_3 + 72x_4 - 18x_5 &= -37. \end{aligned}$$

**25.** Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra $\lambda$

$(a)$

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 10\\ -3x_1 + 5x_2 - x_3 - \lambda(\lambda-1)\, x_4 &= 9 - \lambda\\ 2x_1 - 4x_2 + 5x_3 &= 18\\ 2x_1 + 3x_2 - 4x_3 + \lambda(\lambda-1)\, x_4 &= \lambda - 9 \end{aligned}$$

$(b)$

$$\begin{aligned} x_1 - 4x_2 + 3x_3 &= 4\\ -3x_1 + 14x_2 - 10x_3 - \lambda(\lambda-3)\, x_4 &= -\lambda - 6\\ 3x_1 - 14x_2 + 10x_3 &= 9\\ 2x_1 - 3x_2 + 4x_3 + \lambda(\lambda-3)\, x_4 &= \lambda + 16 \end{aligned}$$

$(c)$

$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= -8\\ -4x_1 + 8x_2 + x_3 - \lambda(\lambda+2)\, x_4 &= 37 - \lambda\\ 2x_1 - 9x_2 + 5x_3 &= -28\\ 3x_1 + 2x_2 - 5x_3 + \lambda(\lambda+2)\, x_4 &= \lambda \end{aligned}$$

**26.** Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra $\lambda$

$(a)$

$$\begin{aligned} -x + 6y + (\lambda+3)z &= 21\\ -x + 3y + 2z &= 9\\ x + 3y + 2\lambda z &= \lambda + 13. \end{aligned}$$

$(b)$

$$\begin{aligned} -x + 8y + (\lambda+4)z &= 29\\ -x + 4y + 3z &= 13\\ x + 4y + (2\lambda-1)z &= \lambda + 16. \end{aligned}$$

$(c)$

$$\begin{aligned} -x + 10y + (\lambda+5)z &= 37\\ -x + 5y + 4z &= 17\\ x + 5y + (2\lambda-2)z &= \lambda + 19. \end{aligned}$$

**27.** Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra $\lambda$

$(a)$

$$\begin{aligned} \lambda x + \ 2y + \ z &= 3 \\ -9x - 2\lambda y + 3z &= \lambda \\ 8x + \ \lambda y + 2z &= 6. \end{aligned}$$

$(b)$

$$\begin{aligned} 2x + (2\lambda - 4)y + (\lambda - 3)z &= 8 \\ 2x + \ (\lambda - 2)y \qquad\qquad &= 5 \\ -3x \qquad\qquad + (\lambda - 3)z &= -3. \end{aligned}$$

$(c)$

$$\begin{aligned} x + \ 2y + \ \lambda z &= 1 \\ 2x + (\lambda + 1)y + (2\lambda + 2)z &= 2 \\ -3x - \ 6y + (4 - 2\lambda)z &= -6. \end{aligned}$$

**28.** Riješiti sistem jednačina

$$\begin{aligned} 2x_1 + 5x_2 - 8x_3 &= 8 \\ 4x_1 + 3x_2 - 9x_3 &= 9 \\ 2x_1 + 3x_2 - 5x_3 &= 7 \\ x_1 + 8x_2 - 7x_3 &= 12. \end{aligned}$$

**29.** Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra $\lambda$

$(a)$

$$\begin{aligned} x_1 \qquad\qquad + \ x_3 + \ x_4 &= 1 \\ 2x_1 + (2 - \lambda)x_2 + 3x_3 + 3x_4 &= 7 - \lambda \\ x_1 + (2 - \lambda)x_2 + \ x_3 + \ x_4 &= 3 - \lambda \ . \end{aligned}$$

$(b)$

$$\begin{aligned} x_1 + \ x_2 + \ x_3 \qquad\qquad &= -1 \\ 2x_1 + 3x_2 + 3x_3 + \ (\lambda - 1)x_4 &= -1 \\ 3x_1 + 4x_2 + 4x_3 + (2\lambda - 2)x_4 &= 2 \ . \end{aligned}$$

# 4 Vektorski prostor

**30.** Dat je skup $\mathcal{B}$ i vektor $u$

$$(a)\ \mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -6 \\ -9 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -6 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} \right\};\ u = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

$$(b)\ \mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\};\ u = \begin{bmatrix} 3 \\ -4 \\ 4 \end{bmatrix}$$

Provjeriti da li je skup $\mathcal{B}$ linearno nezavisan. Objasniti zašto je $\mathcal{B}$ baza vektorskog prostora $\mathbb{R}^3$? Vektor $u$ izraziti kao linearnu kombinaciju vektora iz baze $\mathcal{B}$ (drugim riječima, odrediti koordinate vektora $u$ u odnosu na bazu $\mathcal{B}$).

**31.** Date su dvije baze $\mathcal{B}$ i $\mathcal{B}'$ vektorskog prostora $\mathbb{R}^3$. Vektor $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B}$ ima koordinate

$(a)$ $\begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}$ (gdje su $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \right\}$ i $\mathcal{B}' = \left\{ \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$);

$(b)$ $\begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$ (gdje su $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$ i $\mathcal{B}' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$);

$(c)$ $\begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}$ (gdje su $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ i $\mathcal{B}' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$);

$(d)$ $\begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}$ (gdje su $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$ i $\mathcal{B}' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$).

Odrediti koordinate vektora $v$ u odnosu na bazu $\mathcal{B}'$.

**32.** Odrediti sve vrijednosti parametra $m$ tako da vektori

$(a)$ $\vec{a} = \begin{pmatrix} m-2 \\ 1 \\ 2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} m-2 \\ m-2 \\ 3 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} m-2 & 1 & m-2 \end{pmatrix}^\top$;

$(b)$ $\vec{a} = \begin{pmatrix} m-1 \\ m-1 \\ m-1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ m-1 \\ 1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 2 & 3 & m-1 \end{pmatrix}^\top$;

nisu baza (ne čine bazu) vektorskog prostora $\mathbb{R}^3$. Za najveću dobijenu vrijednost parametra $m$ izraziti vektor $\vec{c}$ kao linearnu kombinaciju vektora $\vec{a}$ i $\vec{b}$.

**33.** Ako je $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ jedna baza vektorskog prostora $\mathbb{R}^3$, dokazati da i vektori $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ također čine bazu prostora $\mathbb{R}^3$ i izraziti vektor $\vec{c}$ preko vektora baze $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ ako su

$(a)$ $\vec{b}_1 = \vec{a}_2 + 3\vec{a}_3$, $\vec{b}_2 = \vec{a}_1 + \vec{a}_2 + 2\vec{a}_3$, $\vec{b}_3 = 2\vec{a}_1 + 2\vec{a}_2 + 6\vec{a}_3$ i $\vec{c} = -\vec{a}_1 + \vec{a}_2 + 2\vec{a}_3$;

$(b)$ $\vec{b}_1 = \vec{a}_1 + 2\vec{a}_2 + 3\vec{a}_3$, $\vec{b}_2 = \vec{a}_1 + \vec{a}_2 + 2\vec{a}_3$, $\vec{b}_3 = 2\vec{a}_1 + \vec{a}_2 + 4\vec{a}_3$ i $\vec{c} = \vec{a}_1 + \vec{a}_2 + \vec{a}_3$.

**34.** Za koje vrijednosti parametra $m$ vektori

$(a)$ $\vec{a} = (2m, 1+m, 1)^\top$, $\vec{b} = (-m, 1, m)^\top$ i $\vec{c} = (m, 1, m-2)^\top$;

$(b)$ $\vec{a} = (m, -m, 1)^\top$, $\vec{b} = (-m, m, 2m+2)^\top$ i $\vec{c} = (m, m+1, 1-m)^\top$;

$(c)$ $\vec{a} = (2m, 1-m, 1)^\top$, $\vec{b} = (-2m, m, 2m+2)^\top$ i $\vec{c} = (m, 1+m, 1-m)^\top$;

čine bazu trodimenzionalnog vektorskog prostora?

**35.** Neka je $\mathcal{B} = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ jedna baza vektorskog prostora $\mathbb{R}^3$. Dokazati da je i skup

$\mathcal{B}' = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ također baza prostora $\mathbb{R}^3$ gdje su

$(a)$ $\vec{b}_1 = 14\vec{a}_1 - \vec{a}_2 + 32\vec{a}_3$, $\vec{b}_2 = 16\vec{a}_1 - \vec{a}_2 + 36\vec{a}_3$ i $\vec{b}_3 = -41\vec{a}_1 + 3\vec{a}_2 - 93\vec{a}_3$.

$(b)$ $\vec{b}_1 = 22\vec{a}_1 + \vec{a}_2 + 39\vec{a}_3$, $\vec{b}_2 = -24\vec{a}_1 - \vec{a}_2 - 43\vec{a}_3$ i $\vec{b}_3 = -2\vec{a}_1 - 3\vec{a}_3$.

$(c)$ $\vec{b}_1 = 15\vec{a}_1 - \vec{a}_2 + 33\vec{a}_3$, $\vec{b}_2 = 3\vec{a}_1 + 6\vec{a}_3$ i $\vec{b}_3 = -29\vec{a}_1 + 2\vec{a}_2 - 63\vec{a}_3$.

Odrediti i koordinate vektora $\vec{a}_2$ u odnosu na bazu $\mathcal{B}'$ (drugim riječima napisati vektor $\vec{a}_2$ kao linearnu kombinaciju vektora iz baze $\mathcal{B}'$).

**36.** Date su dvije baze $\mathcal{B}$ i $\mathcal{B}'$ vektorskog prostora $\mathbb{R}^3$.

$(a)$ Vektor $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B}$ ima koordinate $\begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}$ (gdje su $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \right\}$ i $\mathcal{B}' = \left\{ \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$).

$(b)$ Vektor $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B}$ ima koordinate $\begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$ (gdje su $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$ i $\mathcal{B}' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$).

$(c)$ Vektor $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B}$ ima koordinate $\begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}$ (gdje su $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ i $\mathcal{B}' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$).

Odrediti koordinate vektora $v$ u odnosu na bazu $\mathcal{B}'$.

# 5 Limesi

**37.** Bez upotrebe H'Lopitalovog pravila izračunati limese

$$(a)\ \lim_{x\to 3} \frac{3x^2 - 10x + 3}{2x^2 - 7x + 3}; \quad (b)\ \lim_{x\to -\frac{\pi}{2}} \frac{\cos^2 x}{1 + \sin^3 x}; \quad (c)\ \lim_{x\to 7} \frac{2x^2 - 13x - 7}{-2x^2 + 11x + 21};$$

$$(d)\ \lim_{x\to \frac{\pi}{2}} \frac{1 - \sin^3 x}{\cos^2 x}; \quad (e)\ \lim_{x\to 1} \frac{5x^2 - 3x - 2}{7x^2 - 10x + 3}; \quad (f)\ \lim_{x\to 0} \frac{1 - \cos^3 x}{\sin^2 x};$$

$$(g)\ \lim_{x\to 5} \frac{2x^2 - 11x + 5}{3x^2 - 14x - 5}; \quad (h)\ \lim_{x\to \pi} \frac{\sin^2 x}{1 + \cos^3 x}.$$

**38.** Bez upotrebe H'Lopitalovog pravila izračunati limese

$$(a)\ \lim_{x\to -2} \frac{-5x^2 - 30x - 40}{-3x^2 + 6x + 24}; \quad (b)\ \lim_{x\to -4} \frac{-4x^2 - 12x + 16}{-2x^2 - 10x - 8};$$

$$(c)\ \lim_{x\to -6} \frac{3x^2 + 12x - 36}{2x^2 + 10x - 12}; \quad (d)\ \lim_{x\to -8} \frac{5x^2 + 35x - 40}{-2x^2 - 6x + 80}.$$

**39.** Bez upotrebe H'Lopitalovog pravila izračunati limese

$$(a)\ \lim_{x\to 4}\frac{2-\sqrt{x}}{3-\sqrt{2x+1}}; \qquad (b)\ \lim_{x\to 9}\frac{4-\sqrt{2x-2}}{3-\sqrt{x}}.$$

# 6 Izvodi

**40.** Odrediti prvi izvod funkcije

$$(a)\ y=\ln\frac{x^2-1}{x+1}+\operatorname{arc\,tg}x^2 \qquad (b)\ y=\ln\frac{x}{x-1}+\arcsin x^2 \qquad (c)\ y=\ln\frac{x^2}{x+1}+\operatorname{tg}x^2$$

# 7 Jednačina tangente i normale na krivu

**41.** Odrediti jednačinu tangentne i normale

(*a*) na krivu $x^2+y^2-2x+4y-3=0$;

(*b*) na krivu $x^2+y^2+4x-2y+3=0$;

u tačkama u kojima kriva siječe $x$-osu.

# 8 Ispitivanje funkcija

**42.** Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti funkcija

$$(a)\ y=\frac{(x-3)^3}{(x-4)^2} \qquad (b)\ y=\frac{(x-2)^3}{(x+1)^2}$$

**43.** Odrediti kosu asimptotu sljedećih funkcija

$$(a)\ y=\frac{3x^4-x}{x^3+2}; \qquad (b)\ y=\frac{x^4+1}{x^3-1}; \qquad (c)\ y=\frac{2x^2-3x+4}{x-2}: \qquad (d)\ y=\frac{2x^3+4}{x^2-x+1}.$$

**44.** Odrediti definiciono područje, ekstreme, prevojne tačke, te intervale konveksnosti i konkavnosti funkcije

$$(a)\ y=\frac{3x^2-15x+108}{x-5}; \qquad (b)\ y=\frac{2x^2-6x+2}{x-3}; \qquad (c)\ y=\frac{4x^2+8x+1}{x+2}.$$

**45.** Ispitati i nacrtati grafik sljedećih funkcija

$$(a)\ y=\frac{x-2}{x^2-8x+16}; \qquad (b)\ y=\frac{x-5}{x^2-2x+1};$$

$$(c)\ y=\frac{x-3}{x^2-4x+4}; \qquad (d)\ y=\frac{x-1}{x^2-10x+25};$$

**46.** Ispitati i grafički predstaviti sljedeće funkcije

(*a*) $y=\dfrac{x^3-2}{2x^2}$ (ima greška u rješenju ovog zadataka - prvi integral nije dobar);

(*b*) $y=\dfrac{x^2+10}{x^2+4x+4}$.

**47.** Ispitati funkciju i nacrtati njen grafik $\qquad y=\dfrac{3x^3-1}{(x+1)^3}$.

**48.** Odrediti parametre $a$ i $b$ tako da je prava

$(a)$ $y = x - 4$ kosa asimptota funkcije $y = \dfrac{(ax+b)^4}{x^3}$;

$(b)$ $y = 27x + 9$ kosa asimptota funkcije $y = \dfrac{(ax+b)^3}{x^2}$;

$(c)$ $y = 4x + 4$ kosa asimptota funkcije $y = \dfrac{(ax+b)^2}{x}$;

$(d)$ $y = 64x - 27$ kosa asimptota funkcije $y = \dfrac{a^2x^3 + b^3x^2 + 1}{x^2}$.

**49.** Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti funkcije

$$(a)\ y = \frac{\ln x}{x}; \quad (b)\ y = \frac{1+\ln x}{x^2}; \quad (c)\ y = \frac{1-\ln x}{x^2}; \quad (d)\ y = \frac{1+\ln x}{\ln x};$$
$$(e)\ y = \frac{2+\ln x}{6x^2}; \quad (f)\ y = \frac{3+\ln x}{x}.$$

**50.** Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti funkcije

$$(a)\ y = x^2e^{-\frac{x}{3}}; \quad (b)\ y = xe^{-\frac{1}{x}}; \quad (c)\ y = xe^{-\frac{x^2}{4}}; \quad (d)\ y = xe^{-\frac{x}{2}}.$$

**51.** Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti funkcije

$$(a)\ y = \frac{e^{2x}}{x+1}; \quad (b)\ y = \frac{e^{3x}}{1+e^{-x}}; \quad (c)\ y = \frac{e^{2x}}{1+e^{2x}}; \quad (d)\ y = \frac{e^{3x}}{1-x}.$$

**52.** Ispitati funkciju i nacrtati njen grafik $\qquad y = \dfrac{e^{2x}}{e^{2x} - e^{-x}}$.

**53.** Ispitati i grafički predstaviti sljedeće funkcije

$$(a)\ y = (2x+1)\,e^{-\frac{2}{x}}; \quad (b)\ y = (\frac{1}{2}x - 1)\,e^{-\frac{1}{x}}.$$

**54.** Odrediti definiciono područje, znak te ekstreme funkcije

$$(a)\ y = \ln\frac{x}{x^2-1}; \quad (b)\ y = \ln\frac{x-1}{x^2+1}; \quad (c)\ y = \ln\frac{x^2-1}{x+1}; \quad (d)\ y = \ln\frac{x+1}{x-1}.$$

**55.** Ispitati i grafički predstaviti sljedeće funkcije

$(a)$ $y = \dfrac{\ln^2 x + 1}{x^2}$;

$(b)$ $y = \ln\dfrac{x^2 - 3x + 2}{x^2+1}$.

**56.** Ispitati i grafički predstaviti sljedeće funkcije

$$(a)\ y = \frac{3x^2-1}{(x^2+1)^3}; \quad (b)\ y = \ln(2x^2 - x^4); \quad (c)\ y = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

**57.** Ispitati i grafički predstaviti sljedeće funkcije

$$(a)\ y = \ln(2x - x^3); \quad (b)\ y = \frac{3x-1}{(x^2+1)^2}; \quad (c)\ y = \frac{e^x}{e^x + e^{-x}}.$$

**58.** Ispitati i grafički predstaviti sljedeće funkcije

$$(a)\ y=\frac{3x-1}{(x+1)^3};\qquad (b)\ y=\ln\frac{2-x^2}{x};\qquad (c)\ y=\frac{e^x-e^{-x}}{e^x}.$$

## 9 Ekstremi funkcija dvije promjenjive

**59.** Odrediti stacionarne tačke funkcije

$$(a)\ z=\frac{1}{2}x^2-xy+xy^2-\frac{1}{2}x^2y;\qquad (b)\ z=9x^2-\frac{9}{2}x^2y+6xy^2-12xy;$$

$$(c)\ z=x^2y-\frac{1}{2}xy^2-xy+\frac{1}{2}y^2;\qquad (d)\ z=6x^2y-\frac{9}{2}xy^2-12xy+9y^2.$$

**60.** Naći ekstreme funkcije $z=x^3+3xy^2-15x-12y$.

**61.** Odrediti ekstreme funkcije

$$(a)\ z=x^2+y^3+4x\sqrt{x^3}-3y;\qquad (b)\ z=3\ln\frac{x}{6}+\ln(12-y-x)+2\ln y;$$

$$(c)\ z=x^3+y^2-3x+4\sqrt{y^5};\qquad (d)\ z=2\ln x+\ln(12-x-y)+3\ln\frac{y}{6}.$$

**62.** Naći ekstreme funkcije $z=\frac{1}{3}x^3-2xy+x+3y^2-4y$

## 10 Neodređeni integrali

**63.** Odrediti integrale $(a)\ I=\int\frac{\sin x\cdot\cos x}{e^x}\mathrm{d}x,\ \ (b)\ \int\frac{\sin x+\cos x}{\sin x+2\cos x}\mathrm{d}x.$

**64.** Odrediti integrale

$$(a)\ \int(x^2+2x)\ \cos 2x\,\mathrm{d}x,\qquad (b)\ \int(\frac{3}{2}x^2+3x)\ \sin 3x\,\mathrm{d}x,$$

$$(c)\ \int x\,\mathrm{arc\,tg}\,x\,\mathrm{d}x,\qquad (d)\ \int x\,\mathrm{arc\,ctg}\,x\mathrm{d}x.$$

**65.** Odrediti integrale

$$(a)\ \int\frac{(5x-3)\,dx}{\sqrt{-34+12x-x^2}},\qquad (b)\ \int\frac{(4x+2)\,dx}{\sqrt{-22+10x-x^2}},$$

$$(c)\ \int\frac{(2x-1)\,dx}{\sqrt{-7+6x-x^2}},\qquad (d)\ \int\frac{(3x-7)\,dx}{\sqrt{-33+12x-x^2}}.$$

**66.** Odrediti integrale

$$(a)\ \int\frac{\mathrm{d}x}{3x-4\sqrt{x}},\qquad (b)\ \int\frac{\sqrt{x}\,\mathrm{d}x}{\sqrt[4]{x^3}+4},$$

$$(c)\ \int\frac{\sqrt{x}\,dx}{\sqrt[3]{x^2}+\sqrt[4]{x}},\qquad (d)\ \int\frac{\sqrt[6]{x+1}\,\mathrm{d}x}{\sqrt{x+1}+\sqrt[3]{x+1}}.$$

**67.** Odrediti integrale

$$(a)\ \int x\ln(x-1)\mathrm{d}x,\qquad (b)\ \int\ln(1+x^2)\mathrm{d}x,$$

$$(c)\ \int\ln(x^2-1)\mathrm{d}x,\qquad (d)\ \int(x+1)\ln x\mathrm{d}x.$$

**68.** Odrediti integral

$$(a)\int\frac{7x-17}{x^2-5x+6}\mathrm{d}x \qquad (b)\int\frac{9x-2}{x^2-x-6}\mathrm{d}x \qquad (c)\int\frac{11x+14}{x^2+3x-4}\mathrm{d}x$$

**69.** Odrediti integrale

$$(a)\int\frac{x-1}{\sqrt{-1+4x-x^2}}\mathrm{d}x; \quad (b)\int\frac{4x^2+11x-2}{x^3-3x-2}\mathrm{d}x.$$

**70.** Odrediti integral

$$(a)\int\frac{6x^2-19x+9}{(x-2)(x^2-5x+6)}\mathrm{d}x \qquad (b)\int\frac{8x^2+39x+11}{(x+2)(x^2-x-6)}\mathrm{d}x \qquad (c)\int\frac{8x^2-35x+3}{(x^2+1)(x-7)}\mathrm{d}x$$

**71.** Odrediti integral $\displaystyle\int\frac{x^3-3}{x^4+10x^2+25}\mathrm{d}x.$

# 11 Određeni integral

**72.** Izračunati integrale

$$(a)\int\limits_{-\pi/2}^{\pi/2} x|\cos x|\,\mathrm{d}x, \qquad (b)\int\limits_{-\pi/2}^{\pi/2} e^x|\cos x|\,\mathrm{d}x,$$

$$(c)\int\limits_{0}^{\pi} x|\sin x|\,\mathrm{d}x, \qquad (d)\int\limits_{0}^{\pi} e^x|\sin x|\,\mathrm{d}x.$$

# 12 Primjena određenog integrala

**73.** Primjenom određenog integrala odrediti površinu figure koju ograničava

($a$) $x$-osa zajedno sa linijama $x+3y-3=0$, $x=-3$ i $x=6$;

($b$) $x$-osa zajedno sa linijama $-x-2y+2=0$, $x=-4$ i $x=2$;

($c$) $y$-osa zajedno sa linijama $x+y-1=0$, $y=3$ i $y=-2$.

**74.** Izračunati površinu ravne figure koja je ograničena linijama $y=-x^2$ i $x-y-2=0$.

**75.** Izračunati površinu ravne figure koja je ograničena parabolama

($a$) $y=4-x^2$ i $y=x^2-2x$; ($b$) $y=-x^2-4x$ i $y=x^2+2x$;

($c$) $x=y^2-1$ i $x=-y^2-2y+3$; ($d$) $x=y^2-4y+3$ i $x=-y^2+2y+3$.

**76.** Odrediti površinu figure ograničene

($a$) hiperbolom $xy=4$ i pravom $y=-x+5$.

($b$) parabolom $y=x^2+4x$ i pravom $x-y+4=0$.

($c$) parabolom $4y=8x-x^2$ i pravom $4y=x+6$.

($d$) hiperbolom $xy=6$ i pravom $y=7-x$.

**77.** Odrediti površinu figure ograničene parabolom $4x=8y-y^2$ i pravom $4x=y+6$.

**78.** Primjenom određenog integrala izračunati površinu figure koju ograničavaju linije

(*a*) $x + 2y - 5 = 0$, $2x + y - 7 = 0$ i $y = x + 1$;

(*b*) $-2x - y + 8 = 0$, $-x - 2y + 7 = 0$ i $y = x + 2$;

(*c*) $y + 2x + 7 = 0$, $x + 2y + 5 = 0$ i $y = x - 1$.

**79.** Izračunati površinu ravne figure ograničene parabolom $y = ax^2 + bx$ koja sadrži tačke $A(-3; -3)$ i $B(-1; -3)$ i pravom $x = y - 4$.

# 13 Diferencijalne jednačine. Diferencijalne jednačine prvog reda.

**80.** Provjeriti da li je data funkcija rješenje date diferencijalne jednačine

(*a*) $y = \sqrt{x}$, $2yy' = 1$; (*b*) $\ln x \ln y = c$, $y \ln y \, \mathrm{d}x + x \ln x \, \mathrm{d}y = 0$;

(*c*) $s = -t - \frac{1}{2}\sin 2t$, $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + \operatorname{tg} t \frac{\mathrm{d}s}{\mathrm{d}t} = \sin 2t$.

**81.** Ako znamo opšte rješenje od $4x^2 + y^2 = C^2$ - neke diferencijalne jednačine prvog reda, odrediti i grafički prikazati integralne krive (parcijalne integrale), koje prolaze kroz tačke $B_1(-1; 0)$, $B_2(0; -3)$ i $B_3(2; 0)$.

**82.** Odrediti tip diferencijalne jednačine:

(*a*) $yy' + xe^y = 0$; (*b*) $y + xy' = 4\sqrt{y'}$; (*c*) $y' - y\operatorname{tg} x + 2\sin x - 1 = 0$;

(*d*) $xy' - y = (x + y)\ln\frac{x + y}{x}$; (*e*) $xy' = y - xy\sin x$; (*f*) $(x^2 + 1)y' - xy^2 = xy(x^2 y - 1)$.

## 13.1 Diferencijalne jednačine sa razdvojenim promjenjivim.

Ove jednačine se mogu svesti na jedan od sljedećih oblika

$$y' = f(x)g(y)$$

ili

$$\frac{f_1(x)}{\varphi_1(x)}\mathrm{d}x + \frac{f_2(x)}{\varphi_2(x)}\mathrm{d}x = 0.$$

Poslije razdvajanja varijabli će svaki član jednakosti zavisiti samo od jedne varijable, pa ćemo opšte rješenje dobiti tako što ćemo integrirati svaki član posebno.

**83.** Odrediti opšte rješenje sljedećih diferencijalnih jednačina:

(*a*) $(x + 1)^3 \mathrm{d}y - (y - 2)^2 \mathrm{d}x = 0$ (*b*) $\frac{1}{\cos^2 x \, \cos y}\mathrm{d}x = -\operatorname{ctg} x \, \sin y \mathrm{d}y$

(*c*) $(\sqrt{xy} + \sqrt{x})y' - y = 0$ (*d*) $2^{x+y} + 3^{x-2y}y' = 0$.

**84.** Odrediti partikularno rješenje diferencijalne jednačine koji zadovoljavaju inicijalni uslov:

(*a*) $y\mathrm{d}x + \operatorname{ctg} x \mathrm{d}y = 0$, $y(\frac{\pi}{3}) = -1$;

(*b*) $s = s' \cos^2 t \ln s$, $s(\pi) = 1$.

**85.** Odrediti opšte rješenje sljedećih diferencijalnih jednačina:

$(a)$ $xy' = y - xy\sin x$; $(b)$ $(xy^2 + 3x)\mathrm{d}x + (2x^2y - 5y)\mathrm{d}y = 0$;

$(c)$ $3y'(x^2 - 1) - 2xy = 0$; $(d)$ $y - xy' = a(1 + x^2y')$; $a = const.$

**86.** Odrediti opšte rješenje diferencijalnih jednačina:

$(a)$ $(x^2y + x^2)\mathrm{d}x + (x^4y - y)\mathrm{d}y = 0$; $(b)$ $y' = 2^{2x+y}$.

**87.** Odrediti opšte rješenje sljedećih diferencijalnih jednačina

$(a)$ $x^2(y + 1)\mathrm{d}x + y^2(x - 1)\mathrm{d}y = 0$; $(b)$ $4x\mathrm{d}y - y\mathrm{d}x = x^2\mathrm{d}y$;

$(c)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{4y}{x(y-3)}$.

**88.** Odrediti partikularno rješenje diferencijalne jednačine $(1 + x^3)\mathrm{d}y - x^2y\mathrm{d}x = 0$ koje zadovoljava inicijalni uslov $x = 1$, $y = 2$.

## 13.2 Homogene jednačine prvog reda.

Ove jednačine se mogu svesti na sljedeći oblika

$$y' = f\left(\frac{y}{x}\right).$$

Homogene diferencijalne jednačine rješavamo tako što uvorimo smjenu $\dfrac{y}{x} = u$, iz čega slijedi da je $y = ux$, $y' = u'x + u$.

**89.** Odrediti opšte rješenje sljedećih diferencijalnih jednačina:

$(a)$ $xy' + y = -x$;

$(b)$ $xy' = y(1 + \ln y - \ln x)$ tako da zadovoljava uslov $y(1) = e$;

**90.** Odrediti opšte rješenje diferencijalnih jednačina:

$(a)$ $xy' = xe^{\frac{y}{x}} + y$; $(b)$ $y^3y' + 3xy^2 + 2x^3 = 0$;

$(c)$ $(3y^2 + 3xy + x^2)\mathrm{d}x = (x^2 + 2xy)\mathrm{d}y$; $(d)$ $(5y + 7x)\mathrm{d}y + (8y + 10x)\mathrm{d}x = 0$.

**91.** Odrediti opšte rješenje date diferencijalne jednačine

$(a)$ $y' = \dfrac{x+y}{x-y}$; $(b)$ $y' = \dfrac{y^2}{x^2} - 2$;

$(c)$ $x\,dy - y\,dx = y\,dy$; $(d)$ $y' = \dfrac{2xy}{x^2 - y^2}$.

## 13.3 Diferencijalne jednačine koje se svode na homogene.

Ove jednačine se mogu svesti na sljedeći oblika

$$y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right).$$

Diferencijalne jednačine koje se svode na homogene rješavamo na sljedeći način:

($a$) Ako je $a_1b_2 - a_2b_1 = 0$ tada uvodimo smjenu $a_1x + b_1y = u$ i kao rezultat dobijamo diferencijalnu jednačinu sa razdvojenim promjenjivim.

($b$) Ako je $a_1b_2 - a_2b_1 \neq 0$ tada uvodimo smjenu $x = u + \alpha$, $y = v + \beta$ gdje brojeve $\alpha$ i $\beta$ dobijamo rejšenjem sistema:

$$\begin{aligned} a_1\alpha + b_1\beta + c_1 &= 0 \\ a_2\alpha + b_2\beta + c_2 &= 0 \end{aligned}$$

**92.** Odrediti opšte rješenje diferencijalnih jednačina:

($a$) $(x - 2y + 1)y' = 2x - y + 1$; ($b$) $(2x + y + 1)y' = 4x + 2y + 3$;

($c$) $(2x - 4y + 6)\mathrm{d}x + (x + y - 3)\mathrm{d}y = 0$; ($d$) $(x - y - 2)\mathrm{d}x + (2x - y - 5)\mathrm{d}y = 0$.

**93.** Rješiti diferencijalnu jednačinu

($a$) $(2x - 5y + 3)\mathrm{d}x - (2x + 4y - 6)\mathrm{d}y = 0$;

($b$) $(x - y - 1)\mathrm{d}x + (4y + x - 1)\mathrm{d}y = 0$;

($c$) $(x + y)\mathrm{d}x + (3x + 3y - 4)\mathrm{d}y = 0$.

## 13.4 Linearne diferencijalne jednačina.

Ove jednačine se mogu svesti na sljedeći oblika

$$y' + p(x)y = q(x)$$

gdje su $p(x)$ i $q(x)$ neke funkcije po $x$-u. Rješavamo ih uvođenjem smjene $y = uv$, gdje su $u$ i $v$ dvije pomoćne funkcije, nakon čega dobijamo dvije jednačine sa razdvojenim promjenjivim, u odnosu na svaku od pomočnih funkcija.

**94.** Odrediti opšte rješenje diferencijalnih jednačina:

($a$) $(1 + x^2)y' = x(2y + 1)$; ($b$) $xy' - \dfrac{y}{x+1} = x$, tako da $y(1) = -1$;

($c$) $y' + y\cos x = 0,5\sin 2x$; ($d$) $y' - \dfrac{xy}{1 + x^2} = \dfrac{x\sqrt{1 + x^2}}{x^2 - 2x + 2}$.

**95.** Odrediti opšte rješenje diferencijalnih jednačina:

($a$) $(x^2 + 2x - 2y)\mathrm{d}x - \mathrm{d}y = 0$; ($b$) $y'\cos x - y\sin x = x^3e^{x^2}$, uz uslov $y(0) = 1$.

**96.** Odrediti opšte rješenje diferencijalne jednačine

($a$) $xy' - \dfrac{y}{x+1} = x$ koje zadovoljava uslov $y(1) = 0$;

$(b)$ $y' - y\, tg\, x = \dfrac{1}{cos\, x}$ koje zadovoljava uslov $y(0) = 0$;

$(c)$ $xy' + y - e^x = 0$ koje zadovoljava uslov $y(a) = b$;

$(d)$ $xy' - 3y = x^4 e^x$ koje zadovoljava uslov $y(1) = e$.

**97.** Rješiti diferencijalnu jednačinu $y' - y \operatorname{ctg} x = \sin x$.

**98.** Rješiti diferencijalnu jednačinu

$(a)$ $y' + y \operatorname{tg} x = \cos x$;

$(b)$ $x^2y^2y' + xy^3 = y^2$;

$(c)$ $y' - y\ \sin 2x = e^{\sin^2 x}$.

**99.** Rješiti diferencijalnu jednačinu $(x-2)\dfrac{\mathrm{d}y}{\mathrm{d}x} = y + 2(x-2)^3$.

**100.** Rješiti diferencijalnu jednačinu

$(a)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 4x$;

$(b)$ $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = y + x^3 + 3x^2 - 2x$.

## 13.5 Bernulijeva diferencijalna jednačina.

Jednačina Bernulija

$$y' + p(x)y = y^n q(x)$$

se od linearne jednačine razlikuje samo u desnoj strani, i rješava se na potpuno isti način kao i linearna diferencijalna jednačina - pomoću smjene $y = uv$ ona se također svodi na dvije jednačine sa razdvojenim promjenjivim.

**101.** Riješiti diferencijalne jednačine:

$(a)$ $xy' - x^2\sqrt{y} = 4y$; $(b)$ $y' = xy^3 - y$, tako da prolazi kroz $A(0,1)$;

$(c)$ $(1-x^2)y' = xy + xy^2$; $(d)$ $y' + \dfrac{y}{4x} + y^3 e^{\sqrt{x}} = 0$, ako je $y(1) = 1$.

**102.** Odrediti opšte rješenje diferencijalnih jednačina:

$(a)$ $y' = y^4 \cos x + y \operatorname{tg} x$; $(b)$ $y' = \dfrac{3x^2}{x^3 + y + 1}$.

**103.** Odrediti opšte rješenje diferencijalne jednačine $2x^3y' = 2x^2y - y^3$.

**104.** Rješiti diferencijalnu jednačinu

$(a)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{y}{x} = y^2$;

$(b)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{1}{3}y = e^x y^4$;

$(c)$ $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = xy^3$.

## 13.6 Lagranžova diferencijalna jednačina.

Ove jednačine su oblika

$$y = xf(y') + g(y')$$

Rješavamo ih tako što uvodimo smjenu $y' = p$ ($\mathrm{d}y = p\mathrm{d}x$), poslije čega obično dobijamo linearnu diferencijalnu jednačinu po $x$-u, pa uvodimo novu smjenu $x = uv$.

**105.** Riješiti diferencijalne jednačine:

$(a)$ $y + xy' = 4\sqrt{y'}$;  $(b)$ $y'(2x - y) = y$;

$(c)$ $y = xy' - 2 - y'$, tako da prolazi kroz $A(2, 5)$.

**106.** Odrediti opšte rješenje diferencijalnih jednačina:

$(a)$ $2y + y'(2x + y') = 0$;  $(b)$ $y + \dfrac{1}{y'} = \dfrac{y}{x}$.

## 13.7 Klerova diferencijalna jednačina.

Ove jednačine su oblika

$$y = xy' + f(y')$$

i rješavaju se na potpuno isti način kao i Lagranžove diferencijalne jednačine - uvodimo smjenu $y' = p$ ($\mathrm{d}y = p\mathrm{d}x$)...

**107.** Riješiti diferencijalne jednačine:

$(a)$ $xy' + \sin y' - y = 0$;  $(b)$ $y - xy' - \dfrac{y'^2}{2} = 0$;

$(c)$ $2y - 2xy' = a(\sqrt{1 + (y')^2} - y')$.

**108.** Odrediti opšte rješenje datih diferencijalnih jednačina

$(a)$ $y - xy' - \dfrac{1}{2}y'^2 = 0$;

$(b)$ $y'^2 - xy' + y = 0$;

$(c)$ $(y - y'x)^2 = 1 + y'^2$;

$(d)$ $y = y'x + \sqrt{4 + y'^2}$.

(#) Izračunati determinantu $D = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 4 & 4 \\ -2 & -2 & -2 & 1 \\ 3 & 3 & 6 & x^2-3 \end{vmatrix}$,

a zatim riješiti nejednačinu $D < 2x$.

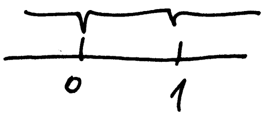
Rj.-upute:

$$D = 2x^2$$

$$2x^2 < 2x$$

$$2x^2 - 2x < 0$$

$$2x(x-1) < 0$$



| | $(-\infty,0)$ | $(0,1)$ | $(1,+\infty)$ |
|---|---|---|---|
| $x$ | $-$ | $+$ | $+$ |
| $x-1$ | $-$ | $-$ | $+$ |
| | $+$ | $-$ | $+$ |

Rješenje nejednačine su svi $x \in (0,1)$.

(#) Izračunati determinantu

$$D = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & 6 & 5 \\ -1 & -1 & 0 & 2 \\ -3 & -3 & -6 & x^2-3 \end{vmatrix},$$

a zatim riješiti nejednačinu $D > 6x$.

Rj: - upute:

$D = 2x^2$

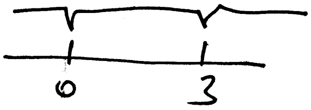
$D > 6x$

$2x^2 > 6x \quad /:2$

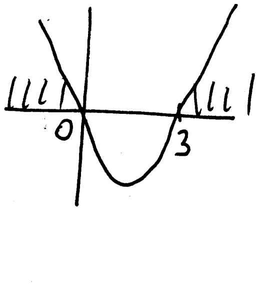
$x^2 > 3x$

$x^2 - 3x > 0$

$x(x-3) > 0$



Rješenje nejednačine su svi $x \in (-\infty, 0) \cup (3+\infty)$,

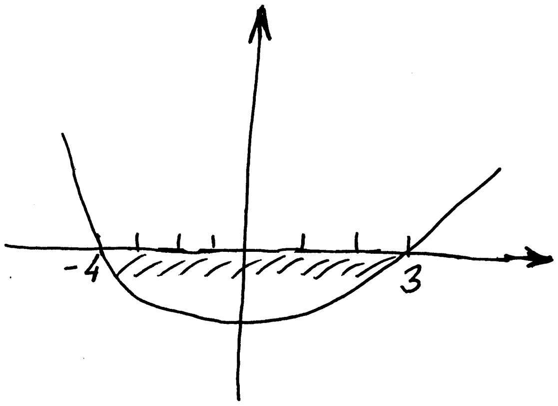


(#) Izračunati determinantu $D = \begin{vmatrix} x^2-8 & 2 & 3 & 1 \\ 4 & 2 & 3 & 1 \\ -4 & -2 & 1 & 0 \\ 5 & 3 & 2 & 1 \end{vmatrix}$,

a zatim riješiti nejednačinu $D < -x$.

Rj.-upute:

$$D = x^2 - 12$$



$$x^2 - 12 < -x$$

$$x^2 + x - 12 < 0$$

$$(x-3)(x+4) < 0$$

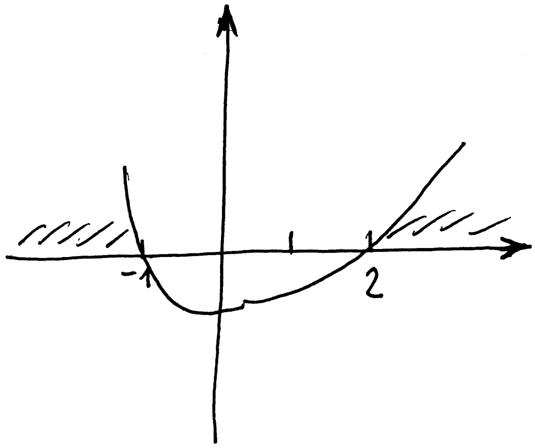
$$x \in (-4, 3)$$

(#) Izračunati determinantu $D = \begin{vmatrix} 2 & 1 & 1 & -1 \\ 4 & x^2-1 & 1 & -1 \\ -4 & -2 & -1 & 0 \\ 5 & 3 & 2 & -1 \end{vmatrix}$

a zatim riješiti nejednačinu $D > x$.

Rj.-upute:

$$D = x^2 - 2$$



$$x^2 - 2 > x$$

$$x^2 - x - 2 > 0$$

$$(x+1)(x-2) > 0$$

$$x \in (-\infty, -1) \cup (2, +\infty)$$

(#) Izračunati determinantu $D = \begin{vmatrix} 1 & -2 & 2 & 1 \\ 3 & -7 & 6 & 2 \\ -4 & 6 & -9 & -6 \\ 5 & -7 & 12 & x^2 \end{vmatrix}$,

a zatim riješiti nejednačinu $D > -2x$.

Rj.-upute

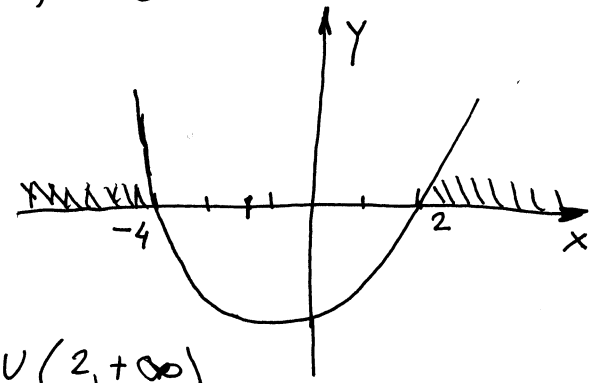
$D = x^2 - 8$

$D > -2x$

$x^2 - 8 > -2x$

$x^2 + 2x - 8 > 0$

$(x-2)(x+4) > 0$



$x \in (-\infty, -4) \cup (2, +\infty)$

(#) Izračunati determinantu $D = \begin{vmatrix} 1 & -2 & 2 & 1 \\ 3 & -7 & 6 & 2 \\ -4 & 6 & x^2-13 & -6 \\ 5 & -7 & 12 & 7 \end{vmatrix}$,

a zatim riješiti nejednačinu $D < 4x$.

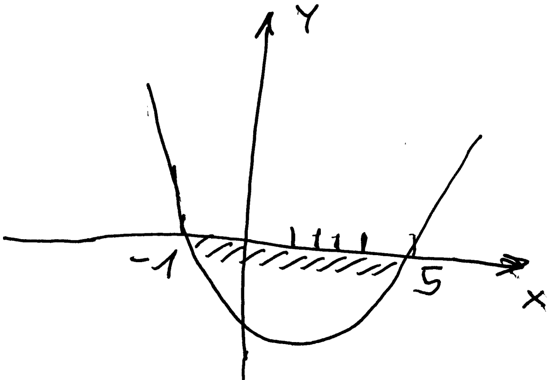
Rj.-upute:

$D = x^2 - 5$

$x^2 - 5 < 4x$

$x^2 - 4x - 5 < 0$

$(x+1)(x-5) < 0$



$x \in (-1, 5)$.

(#) Riješiti matričnu jednačinu $BX = A + I$ ako je

$$A = \begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 7 & -3 \\ 6 & 9 & -1 \end{bmatrix} \quad i \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Rj.-upute:

$BX = A + I \quad / \cdot B^{-1}$ sa lijeve strane

$$X = B^{-1}(A+I)$$

$$\det(B) = 4$$

$$B^{-1} = \frac{1}{\det B} B_{kof}^{T} = \frac{1}{4} \begin{bmatrix} 20 & -6 & -2 \\ -14 & 4 & 2 \\ -6 & 0 & 2 \end{bmatrix}$$

$$A + I = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$X = B^{-1}(A+I) = \frac{1}{4} \begin{bmatrix} 20 & -6 & -2 \\ -14 & 4 & 2 \\ -6 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} =$$

$$= \frac{1}{4} \begin{bmatrix} 18 & 54 & -16 \\ -12 & -38 & 12 \\ -4 & -18 & 8 \end{bmatrix} = \begin{bmatrix} \frac{9}{2} & \frac{27}{2} & -4 \\ -3 & -\frac{19}{2} & 3 \\ -1 & -\frac{9}{2} & 2 \end{bmatrix}$$

traženo rješenje

(#) Riješiti matričnu jednačinu $2I + B\underline{X} = A$ ako je

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & -1 \\ -2 & -3 & 1 \\ -3 & -6 & 4 \end{bmatrix} \quad i \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Rj.-upute:

$$2I + B\underline{X} = A$$

$$B\underline{X} = A - 2I \quad / \cdot B^{-1} \text{ sa lijeve strane}$$

$$\underline{X} = B^{-1}(A-2I)$$

$$\det(B) = 1$$

$$B^{-1} = \frac{1}{\det B} \cdot B_{kof}^{T} = \begin{bmatrix} -6 & -2 & -1 \\ 5 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

$$\underline{X} = B^{-1}(A-2I) = \begin{bmatrix} -6 & -2 & -1 \\ 5 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -10 & 9 \\ -5 & 9 & -7 \\ -3 & 6 & -4 \end{bmatrix}$$

traženo rješenje

(#) Riješiti matričnu jednačinu $-2X = 3AX - I$

ako je $A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 4 & -1 & -1 \end{bmatrix}$ i $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rj.-upute

$$-2X = 3AX - I$$

$$-2X - 3AX = -I$$

$$(-2I - 3A)X = -I \quad / \cdot (-2I - 3A)^{-1} \text{ sa lijeve strane}$$

$$X = (-1)(-2I - 3A)^{-1}$$

$$-2I - 3A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -12 & 3 & 1 \end{bmatrix}$$

$$\det(-2I - 3A) = 1$$

$$(-2I - 3A)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 3 & -3 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} -1 & 0 & 0 \\ -3 & -1 & 0 \\ -3 & 3 & -1 \end{pmatrix}$$

traženo rješenje

(#) Riješiti matričnu jednačinu $-3X = 2AX + I$

ako je $A = \begin{bmatrix} -2 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ i $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Rj - upute

$$-3X = 2AX + I$$

$$-3X - 2AX = I$$

$$(-3I - 2A)X = I \quad / \cdot (-3I - 2A)^{-1} \text{ sa lijeve strane}$$

$$X = (-3I - 2A)^{-1}$$

$$-3I - 2A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(-3I - 2A) = -1$$

$$X = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

traženo rješenje

(#) Riješiti matričnu jednačinu $I + AX = -2X$

ako je $A = \begin{bmatrix} -8 & -2 & 9 \\ -3 & -3 & 4 \\ 1 & 0 & -3 \end{bmatrix}$ i $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rj. - upute

$$I + AX = -2X$$

$$AX + 2X = -I$$

$$(A+2I)X = -I \quad / \cdot (A+2I)^{-1} \text{ sa lijeve strane}$$

$$X = (-1)(A+2I)^{-1}$$

$$A + 2I = \begin{bmatrix} -6 & -2 & 9 \\ -3 & -1 & 4 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\det(A+2I) = 1$$

$$(A+2I)^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -3 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} -1 & 2 & -1 \\ 1 & 3 & 3 \\ -1 & 2 & 0 \end{bmatrix}$$ traženo rješenje

(#) Riješiti matričnu jednačinu $AX + I = -3X$

ako je $A = \begin{bmatrix} 11 & -2 & -13 \\ -6 & -2 & 5 \\ -1 & 0 & -2 \end{bmatrix}$ ; $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rj.: -upute:

$$AX + I = -3X$$

$$AX + 3X = -I$$

$$(A+3I)X = -I \quad /(A+3I)^{-1} \text{ sa lijeve strane}$$

$$X = (A+3I)^{-1} \cdot (-I)$$

$$A+3I = \begin{bmatrix} 14 & -2 & -13 \\ -6 & 1 & 5 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\det(A+3I) = -1$$

$$(A+3I)^{-1} = \begin{bmatrix} -1 & -2 & -3 \\ -1 & -1 & -8 \\ -1 & -2 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 8 \\ 1 & 2 & 2 \end{bmatrix}$$ traženo rješenje

(#) Riješiti matričnu jednačinu

$$\underline{X}\cdot(-2) = \begin{bmatrix}1&0\\0&1\end{bmatrix} + \underline{X}\begin{bmatrix}1&2&3\\-4&-5&-6\end{bmatrix}\begin{bmatrix}-1&-8\\7&1\\-2&-4\end{bmatrix}$$

Rj-upute:

Označimo sa A matricu $A = \begin{bmatrix}1&2&3\\-4&-5&-6\end{bmatrix}\begin{bmatrix}-1&-8\\7&1\\-2&-4\end{bmatrix} = \begin{bmatrix}7&-18\\-19&51\end{bmatrix}$

Imamo

$$\underline{X}\cdot(-2) = I + \underline{X}A$$

$$\underline{X}\cdot(-2) - \underline{X}A = I$$

$$\underline{X}(-2I-A) = I \qquad /\cdot(-2I-A)^{-1} \text{ sa desne strane}$$

$$\underline{X} = (-2I-A)^{-1}$$

Neka je $D = -2I - A = \begin{bmatrix}-2&0\\0&-2\end{bmatrix} - \begin{bmatrix}7&-18\\-19&51\end{bmatrix} = \begin{bmatrix}-9&18\\19&-53\end{bmatrix}$

Kako je

$$D^{-1} = \frac{1}{\det D}D_{kof}^{T} = \frac{1}{\det D}D_{adj}$$

to je $\det(D) = 135$

$$D^{-1} = \begin{bmatrix}-\frac{53}{135} & -\frac{2}{15}\\ -\frac{19}{135} & -\frac{1}{15}\end{bmatrix}$$

pa je $\underline{X} = \begin{bmatrix}-\frac{53}{135} & -\frac{2}{15}\\ -\frac{19}{135} & -\frac{1}{15}\end{bmatrix}$ traženo rješenje

(#) Riješiti matričnu jednačinu

$$X \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ -7 & -1 \\ 2 & 4 \end{bmatrix} = X \cdot (-1) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rj.-upute

Označimo sa $B$ matricu

$$B = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ -7 & -1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -18 \\ -19 & 51 \end{bmatrix}$$

Sad imamo

$$X B = X \cdot (-1) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X B - X \cdot (-1) = I$$

$$X (B + I) = I \quad / \cdot (B+I)^{-1} \text{ sa desne strane}$$

$$X = (B + I)^{-1}$$

Ako sa $A$ označimo matricu $A = B + I$ imamo

$$A = \begin{bmatrix} 8 & -18 \\ -19 & 52 \end{bmatrix} \qquad A^{-1} = \frac{1}{\det A} A_{kof}^{T} = \frac{1}{\det A} A_{adj}$$

$$\det(A) = 74$$

$$A^{-1} = \begin{bmatrix} \frac{26}{37} & \frac{9}{37} \\ \frac{19}{74} & \frac{4}{37} \end{bmatrix}$$

to je $X = \begin{bmatrix} \frac{26}{37} & \frac{9}{37} \\ \frac{19}{74} & \frac{4}{37} \end{bmatrix}$ traženo rješenje

Ⓣ Riješiti matričnu jednačinu

$$\begin{bmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -8 \\ 7 & -1 \\ 2 & -4 \end{bmatrix} \underline{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\underline{X}$$

Rj. -upute

Označimo sa A matricu

$$A = \begin{bmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -8 \\ 7 & -1 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ -19 & -51 \end{bmatrix}$$

Sad imamo

$$A\underline{X} = I - 2\underline{X}$$

$$A\underline{X} + 2\underline{X} = I$$

$$(A+2I)\underline{X} = I \quad /(A+2I)^{-1} \text{ sa lijeve strane}$$

$$\underline{X} = (A+2I)^{-1}$$

Ako matricu $A+2I$ označimo sa $B$ imamo $B = \begin{bmatrix} 9 & 18 \\ -19 & -49 \end{bmatrix}$

$$B^{-1} = \frac{1}{\det B} \cdot B_{kof}^{T} = \frac{1}{\det B} \cdot B_{adj}$$

$$\det(B) = -99$$

$$B^{-1} = \begin{bmatrix} \frac{49}{99} & \frac{2}{11} \\ -\frac{19}{99} & -\frac{1}{11} \end{bmatrix}$$

to je $\underline{X} = \begin{bmatrix} \frac{49}{99} & \frac{2}{11} \\ -\frac{19}{99} & -\frac{1}{11} \end{bmatrix}$ traženo rješenje

(#) Riješiti matričnu jednačinu

$$3X = \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\begin{bmatrix}-1 & 8\\ -7 & 1\\ 2 & 4\end{bmatrix}X + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Rj. - upute

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\begin{bmatrix}-1 & 8\\ -7 & 1\\ 2 & 4\end{bmatrix} = \begin{bmatrix}-9 & 22\\ -27 & 61\end{bmatrix}$$

Ako ovu matricu označimo sa A imamo

$$3X = AX + I$$

$$3X - AX = I$$

$$(3I - A)X = I \quad /(3I-A)^{-1} \text{ sa lijeve strane}$$

$$X = (3I-A)^{-1}$$

$$3I - A = \begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix} - \begin{bmatrix}-9 & 22\\ -27 & 61\end{bmatrix} = \begin{bmatrix}12 & -22\\ 27 & -58\end{bmatrix}$$

$$\det(3I-A) = -102$$

Ako sa C označimo matricu $3I - A$ znamo

$$C^{-1} = \frac{1}{\det C}\cdot C_{kof}^{T} = \frac{1}{\det C}\cdot C_{adj}$$

$$C = \begin{bmatrix}12 & -22\\ 27 & -58\end{bmatrix}$$

$$\det C = -102$$

$$C^{-1} = \begin{bmatrix}\frac{29}{51} & -\frac{11}{51}\\ \frac{9}{34} & -\frac{2}{17}\end{bmatrix}$$

to je $X = \begin{bmatrix}\frac{29}{51} & -\frac{11}{51}\\ \frac{9}{34} & -\frac{2}{17}\end{bmatrix}$ traženo rješenje

(#) Riješiti matričnu jednačinu $CXA+XB=A$ ako sa

$$A=\begin{bmatrix}0 & -1 & 2\\ 2 & -1 & -3\\ -1 & 1 & 1\end{bmatrix},\quad B=\begin{bmatrix}0 & -3 & 6\\ 6 & -3 & -9\\ -3 & 3 & 3\end{bmatrix}\ \text{i}\ C=\begin{bmatrix}-1 & 3 & 5\\ 1 & -1 & 4\\ 1 & 1 & -1\end{bmatrix}.$$

Rj.-upute

$CXA+XB=A$ $\quad$ $/A^{-1}$ sa desne strane

$$CXAA^{-1}+XBA^{-1}=AA^{-1}$$

$$CX+XBA^{-1}=I$$

Izračunajmo $BA^{-1}$. $\quad A^{-1}=\begin{bmatrix}2 & 3 & 5\\ 1 & 2 & 4\\ 1 & 1 & 2\end{bmatrix}$

$$BA^{-1}=\begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}=3I$$

Prema tome $\quad CX+X\underbrace{BA^{-1}}_{3I}=I$

Kako je $X\cdot 3I=3I\cdot X$ to je

$(C+3I)X=I$ $\quad$ $/\cdot(C+3I)^{-1}$ sa lijeve strane

$$X=(C+3I)^{-1}$$

$$C+3I=\begin{bmatrix}2 & 3 & 5\\ 1 & 2 & 4\\ 1 & 1 & 2\end{bmatrix}=A^{-1}\quad\Rightarrow\quad X=\begin{bmatrix}0 & -1 & 2\\ 2 & -1 & -3\\ -1 & 1 & 1\end{bmatrix}$$

traženo rješenje

(#) Riješiti matričnu jednačinu $CXB+AX=C$

ako su $A=\begin{bmatrix} 0 & -2 & -4 \\ -4 & -2 & -6 \\ -2 & 0 & -2 \end{bmatrix}$, $B=\begin{bmatrix} -1 & -1 & 1 \\ 1 & -4 & 4 \\ -1 & 1 & -4 \end{bmatrix}$ i $C=\begin{bmatrix} 0 & -1 & -2 \\ -2 & -1 & -3 \\ -1 & 0 & -1 \end{bmatrix}$

Rj. -upute:

$CXB+AX=C \quad /\cdot C^{-1}$ sa lijeve strane

$$C^{-1}CXB+C^{-1}AX=C^{-1}C$$

$$XB+C^{-1}AX=I$$

Izračunajmo $C^{-1}A$.

$$C^{-1}=\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ -1 & 1 & -2 \end{bmatrix}$$

$$C^{-1}A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}=2I$$

$$\boxed{2I\cdot X=X\cdot 2I}$$

Prema tome

$$XB+\underbrace{C^{-1}A}_{2I}X=I$$

$$XB+X\cdot 2I=I$$

$$X(B+2I)=I \quad /\cdot(B+2I)^{-1} \text{ sa desne strane}$$

$$X=\underbrace{(B+2I)^{-1}}_{D}$$

$$D=\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ -1 & 1 & -2 \end{bmatrix}$$

Primjetimo da su matrice $C^{-1}$ i $D$ podudarne. Prema tome

$$X=\begin{bmatrix} 0 & -1 & -2 \\ -2 & -1 & -3 \\ -1 & 0 & -1 \end{bmatrix}$$ traženo rješenje.

(#) Riješit matričnu jednačinu $AXC + XB = C$ ako su

$$A = \begin{bmatrix} 0 & 3 & 5 \\ 1 & 0 & 4 \\ 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ -1 & 1 & -2 \end{bmatrix} \quad i \quad C = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & -1 & 2 \\ -1/2 & 1/2 & -1 \end{bmatrix}.$$

Rj. -upute:

$AXC + XB = C$ $\quad / C^{-1}$ sa desne strane

$$AX\underbrace{CC^{-1}}_{=I} + XBC^{-1} = \underbrace{CC^{-1}}_{=I}$$

$$AX + XBC^{-1} = I$$

Izračunajmo $BC^{-1}$.

$$C^{-1} = \begin{bmatrix} 0 & -2 & -4 \\ -4 & -2 & -6 \\ -2 & 0 & -2 \end{bmatrix}$$

$$BC^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2I$$

Prema tome $\quad AX + X\underbrace{BC^{-1}}_{2I} = I$

$$AX + 2X = I$$

$(A + 2I)X = I$ $\quad / (A+2I)^{-1}$ sa lijeve strane

$$X = (A + 2I)^{-1}$$

$$D = A + 2I = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -1 & -3 \\ -1 & 1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -1 & -3 \\ -1 & 1 & 1 \end{bmatrix}$$ traženo rješenje

(#) Riješiti matričnu jednačinu $AXB^{-1} = 2XB^{-1} - I$ ako su

$$A = \begin{bmatrix} 6 & 7 & 4 \\ 2 & 6 & 2 \\ 2 & 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix}.$$

Rj. -upute:

$$AXB^{-1} = 2XB^{-1} - I$$

$$AXB^{-1} - 2XB^{-1} = -I$$

$$(AX - 2X)B^{-1} = -I \quad /\cdot B \text{ sa desne strane}$$

$$AX - 2X = -B$$

$$(A - 2I)X = -B \quad /\cdot (A-2I)^{-1} \text{ sa lijeve strane}$$

$$X = (A-2I)^{-1} \cdot (-B)$$

Označimo sa $C = A - 2I$. Tada je $C = \begin{bmatrix} 4 & 7 & 4 \\ 2 & 4 & 2 \\ 2 & 4 & 3 \end{bmatrix}$

$$\det(C) = 2$$

$$C^{-1} = \begin{bmatrix} 2 & -\frac{5}{2} & -1 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$X = C^{-1} \cdot (-B) = \begin{bmatrix} 4 & 4 & -8 \\ -3 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$ traženo rješenje

(#) Riješiti matričnu jednačinu $A^{-1}XB = 2A^{-1}X + I$ ako su

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix} \quad i \quad B = \begin{bmatrix} 6 & 7 & 4 \\ 2 & 6 & 2 \\ 2 & 4 & 5 \end{bmatrix}$$

Rj.-upute:

$$A^{-1}XB = 2A^{-1}X + I$$

$$A^{-1}XB - 2A^{-1}X = I$$

$$A^{-1}(XB - 2X) = I \quad / \cdot A \text{ sa lijeve strane}$$

$$XB - 2X = A$$

$$X(B - 2I) = A \quad / \cdot (B-2I)^{-1} \text{ sa desne strane}$$

$$X = A \cdot (B - 2I)^{-1}$$

Označimo sa $C = B - 2I$. Tada

$$C = \begin{bmatrix} 4 & 7 & 4 \\ 2 & 4 & 2 \\ 2 & 4 & 3 \end{bmatrix} \qquad \det(C) = 2$$

$$C^{-1} = \begin{bmatrix} 2 & -\frac{5}{2} & -1 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$X = A \cdot C^{-1} = \begin{bmatrix} 0 & -\frac{11}{2} & 6 \\ 2 & -3 & 0 \\ -1 & \frac{5}{2} & 0 \end{bmatrix}$$

traženo rješenje

(#) Rješiti sistem linearnih jednačina

$$\begin{aligned} x - y + z &= 3 \\ x - y - z &= 4 \\ x + y - z &= 5 \\ x + y + z &= 6 \end{aligned}$$

Rj.-upute:

Rješimo sistem Kroneker-Kapelijevom metodom

$$\bar{A} = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 1 & -1 & -1 & 4 \\ 1 & 1 & -1 & 5 \\ 1 & 1 & 1 & 6 \end{array}\right] \sim \ldots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \operatorname{rang}(A) < \operatorname{rang}(\bar{A})$$

Dati sistem nema rješenje.

(#) Rješiti sistem linearnih jednačina

$$
\begin{aligned}
x - y + z &= 2\\
x - y - z &= 3\\
x + y - z &= 4\\
x + y + z &= 5
\end{aligned}
$$

Rj.-upute:

Rješimo sistem Kroneker-Kapelijevom metodom

$$\bar{A} = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2\\ 1 & -1 & -1 & 3\\ 1 & 1 & -1 & 4\\ 1 & 1 & 1 & 5 \end{array}\right] \sim \ldots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \quad \text{rang}(A) < \text{rang}(\bar{A})$$

Dati sistem nema rješenje.

Ⓣ Riješiti sistem linearnih jednačina

$$
\begin{aligned}
x - y + z &= 1 \\
x - y - z &= 2 \\
x + y - z &= 3 \\
x + y + z &= 4
\end{aligned}
$$

Rj. - upute:

Riješimo sistem Kronecker-Kapelijevom metodom

$$\bar{A} = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 2 \\ 1 & 1 & -1 & 3 \\ 1 & 1 & 1 & 4 \end{array}\right] \sim \ldots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \quad \text{rang}(A) < \text{rang}(\bar{A})$$

Dati sistem nema rješenja

(#) Riješiti sistem jednačina

$$\begin{aligned} x_1-2x_2+x_3+x_4+x_5&=0\\ -x_1+3x_2-3x_3-3x_4-3x_5&=-2\\ 3x_1-6x_2+4x_3+4x_4+4x_5&=3 \end{aligned}$$

Rj.-uputa:

Sistem ćemo riješiti Kroneker-Kapelijevom metodom

$$\bar{A}=[A\,|\,b]=\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 1 & 1 & 0\\ -1 & 3 & -3 & -3 & -3 & -2\\ 3 & -6 & 4 & 4 & 4 & 3 \end{array}\right]\sim \dots \sim$$

$$\sim\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 5\\ 0 & 1 & 0 & 0 & 0 & 4\\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right]$$

$\operatorname{rang}\bar{A}=\operatorname{rang}A=3<5=$ broj nepoznatih

Prema Kroneker-Kapelijevoj metodi sistem ima ∞ mnogo rješenja i dvije promjenjive uzimamo proizvoljno npr. prema dobijenom rezultatu $x_4=s$, $x_5=t$.

$x_1=5$, $x_2=4$, $x_3=3-s-t$, $x_4=s$ $x_5=t$.

(#) Riješiti sistem jednačina

$$
\begin{aligned}
x_1+2x_2-x_3-x_4-x_5&=10\\
x_1+3x_2+x_3+x_4+x_5&=20\\
-3x_1-6x_2+4x_3+4x_4+4x_5&=-27
\end{aligned}
$$

Rj.-uputa:

Sistem ćemo riješiti Kroneker-Kapelijevom metodom

$$\bar{A}=[A\,|\,b]=\left[\begin{array}{ccccc|c} 1 & 2 & -1 & -1 & -1 & 10\\ 1 & 3 & 1 & 1 & 1 & 20\\ -3 & -6 & 4 & 4 & 4 & -27 \end{array}\right]\sim\ldots\sim$$

$$\sim\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 5\\ 0 & 1 & 0 & 0 & 0 & 4\\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right]$$

$\text{rang}(A)=\text{rang}(\bar{A})=3<5=\text{broj nepoznatih}$

Prema Kroneker-Kapelijevoj metodi sistem ima $\infty$ mnogo rješenja i dvije promjenjive uzimamo proizvoljno.

$x_1=5,\ x_2=4,\ x_3=3-s-t,\ x_4=t,\ x_5=s.$

(#) Riješiti sistem jednačina

$$\begin{aligned} x_1+2x_2-4x_3+16x_4-4x_5&=-8\\ -2x_1-3x_2+5x_3-20x_4+5x_5&=7\\ 4x_1+9x_2-18x_3+72x_4-18x_5&=-37 \end{aligned}$$

Rj.-upute:

Sistem ćemo riješiti Kroneker-Kapelijevom metodom

$$\bar{A}=[A\,|\,b]=\left[\begin{array}{ccccc|c} 1 & 2 & -4 & 16 & -4 & -8\\ -2 & -3 & 5 & -20 & 5 & 7\\ 4 & 9 & -18 & 72 & -18 & -37 \end{array}\right]\begin{array}{l} II_v+I_v\cdot 2\\ \sim\\ III_v+I_v\cdot(-4) \end{array}$$

$$\ldots\sim\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 2\\ 0 & 1 & 0 & 0 & 0 & 3\\ 0 & 0 & 1 & -4 & 1 & 4 \end{array}\right]$$

$$\Rightarrow \left.\begin{array}{l} \text{rang}(A)=3\\ \text{rang}(\bar{A})=3\\ \text{broj nepoznatih}=5 \end{array}\right\}\Rightarrow$$

sistem ima beskonačno mnogo rješenja i dvije promjenjive uzimamo proizvoljno npr. $x_4=s$, $x_5=t$

$$\begin{aligned} x_1&=2\\ x_2&=3\\ x_3&=4+4s-t\\ x_4&=s\\ x_5&=t\\ s,t&\in\mathbb{R} \end{aligned}$$

(#) Riješiti sistem jednačina

$$\begin{aligned} x_1+2x_2-4x_3-4x_4-16x_5&=-9\\ 2x_1+5x_2-11x_3-11x_4-44x_5&=-29\\ -4x_1-7x_2+14x_3+14x_4+56x_5&=30 \end{aligned}$$

Rj.-upute:

Sistem ćemo riješiti Kronecker-Kapelijevom metodom

$$\bar{A}=\left[A\,\vdots\,b\right]=\left[\begin{array}{ccccc|c} 1 & 2 & -4 & -4 & -16 & -9\\ 2 & 5 & -11 & -11 & -44 & -29\\ -4 & -7 & 14 & 14 & 56 & 30 \end{array}\right]\quad \begin{array}{l} II_v+I_v\cdot(-2)\\ III_v+I_v\cdot 4 \end{array}$$

$$\dots\sim\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 3\\ 0 & 1 & 0 & 0 & 0 & 4\\ 0 & 0 & 1 & 1 & 4 & 5 \end{array}\right].$$

$$\Rightarrow\left.\begin{array}{l} \text{rang}(A)=3\\ \text{rang}(\bar{A})=3\\ \text{broj nepoznatih}=5 \end{array}\right\}\Rightarrow$$

sistem ima beskonačno mnogo rješenja i dvije promjenjive uzimamo proizvoljno npr. $x_4=s$, $x_5=t$

$x_1=3$

$x_2=4$

$x_3=5-s-4t$

$x_4=s$

$x_5=t$

$s,t\in\mathbb{R}$

(#) Riješiti sistem jednačina

$$\begin{aligned} x_1 + 2x_2 - 4x_3 - 8x_4 - 12x_5 &= -11 \\ -2x_1 - 3x_2 + 5x_3 + 10x_4 + 15x_5 &= 7 \\ -3x_1 - 5x_2 + 10x_3 + 20x_4 + 30x_5 &= 25 \end{aligned}$$

Rj.-upute:

Sistem ćemo riješiti Kroneker-Kapelij-evom metodom

$$\bar{A} = [A \mid b] = \left[\begin{array}{ccccc|c} 1 & 2 & -4 & -8 & -12 & -11 \\ -2 & -3 & 5 & 10 & 15 & 7 \\ -3 & -5 & 10 & 20 & 30 & 25 \end{array}\right] \begin{array}{l} \\ II_v + I_v \cdot 2 \\ \sim \\ III_v + I_v \cdot 3 \end{array}$$

$$\ldots \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 2 & 3 & 7 \end{array}\right]$$

$$\Rightarrow \left.\begin{array}{l} \text{rang}(A) = 3 \\ \text{rang}(\bar{A}) = 3 \\ \text{broj nepoznatih} = 5 \end{array}\right\} \Rightarrow$$

sistem ima beskonačno mnogo rješenja i dvije promjenjive uzimamo proizvoljno

npr. $x_4 = s$, $x_5 = t$

$x_1 = 5$

$x_2 = 6$

$x_3 = 7 - 2s - 3t$

$x_4 = s$

$x_5 = t$

$s, t \in \mathbb{R}$

(#) Riješiti sistem jednačina

$$\begin{aligned} x_1 + 2x_2 - 4x_3 + 8x_4 + 12x_5 &= -10 \\ 3x_1 + 7x_2 - 15x_3 + 30x_4 + 45x_5 &= -43 \\ -2x_1 - 3x_2 + 6x_3 - 12x_4 - 18x_5 &= 13 \end{aligned}$$

Rj.-upute:

Sistem ćemo riješiti Kroneker-Kapelijevom metodom

$$\bar{A} = [A \mid b] = \left[\begin{array}{ccccc|c} 1 & 2 & -4 & 8 & 12 & -10 \\ 3 & 7 & -15 & 30 & 45 & -43 \\ -2 & -3 & 6 & -12 & -18 & 13 \end{array}\right] \begin{array}{l} II_v + I_v \cdot (-3) \\ \sim \\ III_v + I_v \cdot 2 \end{array}$$

$$\ldots \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & -2 & -3 & 6 \end{array}\right]$$

$$\Rightarrow \left.\begin{array}{l} \text{rang}(A) = 3 \\ \text{rang}(\bar{A}) = 3 \\ \text{broj nepoznatih} = 5 \end{array}\right\} \Rightarrow$$

sistem ima beskonačno mnogo rješenja i dvije promjenjive uzimamo proizvoljno npr. $x_4 = s$, $x_5 = t$
$s, t \in \mathbb{R}$

$x_1 = 4$

$x_2 = 5$

$x_3 = 6 + 2s + 3t$

$x_4 = s$

$x_5 = t$

$s, t \in \mathbb{R}$

(#) Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra $\lambda$

$$\begin{aligned} x_1 - x_2 + 2x_3 \qquad\qquad\qquad\qquad &= 10 \\ -3x_1 + 5x_2 - x_3 - \lambda(\lambda-1)x_4 &= 9-\lambda \\ 2x_1 - 4x_2 + 5x_3 \qquad\qquad\qquad\qquad &= 18 \\ 2x_1 + 3x_2 - 4x_3 + \lambda(\lambda-1)x_4 &= \lambda-9 \end{aligned}$$

Rj.

Sistem ćemo riješiti Kroneker-Kapelijevom metodom

$$\bar{A} = \left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 10 \\ -3 & 5 & -1 & -\lambda(\lambda-1) & 9-\lambda \\ 2 & -4 & 5 & 0 & 18 \\ 2 & 3 & -4 & \lambda(\lambda-1) & \lambda-9 \end{array}\right] \overset{II_v + IV_v}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 10 \\ -1 & 8 & -5 & 0 & 0 \\ 2 & -4 & 5 & 0 & 18 \\ 2 & 3 & -4 & \lambda(\lambda-1) & \lambda-9 \end{array}\right] \sim$$

$$\sim \ldots \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & \lambda(\lambda-1) & \lambda-1 \end{array}\right]$$

Diskusija

1° $\lambda = 1 \Rightarrow \operatorname{rang} A = \operatorname{rang} \bar{A} = 3 < 4 \xRightarrow{\text{Kron.-Kap.}}$ sistem ima $\infty$ mnogo rješenja i jednu prom. uzimamo proizvoljno

rješenje sist. je $(x_1, x_2, x_3, x_4) = (2, 4, 6, s)$, $s \in \mathbb{R}$

2° $\lambda = 0 \Rightarrow \left.\begin{array}{l} \operatorname{rang} A = 3 \\ \operatorname{rang} \bar{A} = 4 \end{array}\right\} \xRightarrow{\text{Kron.-Kap.}}$ sistem nema rješ.

3° $\lambda \neq 1$ i $\lambda \neq 0$ $\operatorname{rang} A = \operatorname{rang} \bar{A} = 4 \xRightarrow{\text{Kron.-Kap.}}$ sistem ima jedinstveno rješenje

rješ. sist. je $(x_1, x_2, x_3, x_4) = (2, 4, 6, \frac{1}{\lambda})$

(#) Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra λ

$$\begin{aligned}
x_1-4x_2+3x_3 &= 4\\
-3x_1+14x_2-10x_3-\lambda(\lambda-3)x_4 &= -\lambda-6\\
3x_1-14x_2+10x_3 &= 9\\
2x_1-3x_2+4x_3+\lambda(\lambda-3)x_4 &= \lambda+16
\end{aligned}$$

Rj.-upute:

Sistem ćemo riješiti Kroneker-Kapelijevom metodom

$$\bar{A}=\left[\begin{array}{cccc|c} 1 & -4 & 3 & 0 & 4\\ -3 & 14 & -10 & -\lambda(\lambda-3) & -\lambda-6\\ 3 & -14 & 10 & 0 & 9\\ 2 & -3 & 4 & \lambda(\lambda-3) & \lambda+16\end{array}\right] \overset{II_v+IV_v}{\sim} \left[\begin{array}{cccc|c} 1 & -4 & 3 & 0 & 4\\ -1 & 11 & -6 & 0 & 10\\ 3 & -14 & 10 & 0 & 9\\ 2 & -3 & 4 & \lambda(\lambda-3) & \lambda+16\end{array}\right]$$

$$\sim \ldots \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3\\ 0 & 1 & 0 & 0 & 5\\ 0 & 0 & 1 & 0 & 7\\ 0 & 0 & 0 & \lambda(\lambda-3) & \lambda-3\end{array}\right]$$

Diskusija

1° $\lambda=3 \Rightarrow \operatorname{rang}\bar{A}=\operatorname{rang}A=3<4 \xRightarrow{\text{Kron.-Kap.}}$ sistem ima ∞ mnogo rješenja i jednu promjenj. uzim. proizvoljno

rješenje sistema je $(x_1,x_2,x_3,x_4)=(3,5,7,s),\ s\in\mathbb{R}$

2° $\lambda=0 \Rightarrow \left.\begin{array}{l}\operatorname{rang}A=3\\ \operatorname{rang}\bar{A}=4\end{array}\right\} \xRightarrow{\text{Kron.-Kap.}}$ sistem nema rješ.

3° $\lambda\neq-3$ i $\lambda\neq0$ $\quad \operatorname{rang}A=\operatorname{rang}\bar{A}=4 \xRightarrow{\text{Kron.-Kap.}}$ sistem ima jedinstveno rješenje

rješ. sist. $(x_1,x_2,x_3,x_4)=(3,5,7,\frac{1}{\lambda})$

(#) Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra $\lambda$

$$\begin{aligned}
x_1 - 3x_2 + 2x_3 &= -8\\
-4x_1 + 8x_2 + x_3 - \lambda(\lambda+2)x_4 &= 37-\lambda\\
2x_1 - 9x_2 + 5x_3 &= -28\\
3x_1 + 2x_2 - 5x_3 + \lambda(\lambda+2)x_4 &= \lambda
\end{aligned}$$

Rj. - upute:

Sistem ćemo riješiti Kroneker-Kapelijevom metodom

$$\bar{A} = \left[\begin{array}{cccc|c} 1 & -3 & 2 & 0 & -8\\ -4 & 8 & 1 & -\lambda(\lambda+2) & 37-\lambda\\ 2 & -9 & 5 & 0 & -28\\ 3 & 2 & -5 & \lambda(\lambda+2) & \lambda \end{array}\right] \overset{II_v + IV_v}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 2 & 0 & -8\\ -1 & 10 & -4 & 0 & 37\\ 2 & -9 & 5 & 0 & -28\\ 3 & 2 & -5 & \lambda(\lambda+2) & \lambda \end{array}\right]$$

$$\sim \dots \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 5\\ 0 & 0 & 1 & 0 & 3\\ 0 & 0 & 0 & \lambda(\lambda+2) & \lambda+2 \end{array}\right]$$

Diskusija

1° $\lambda = -2 \Rightarrow \text{rang}\, A = \text{rang}\, \bar{A} = 3 < 4 \xRightarrow{\text{Kron.-Kap.}}$ sistem ima $\infty$ mnogo rješenja i 1 promj. u njim proizvoljno

rješenje sistema je $(x_1, x_2, x_3, x_4) = (1, 5, 3, s)$, $s \in \mathbb{R}$

2° $\lambda = 0 \Rightarrow \begin{array}{l}\text{rang}\, A = 3\\ \text{rang}\, \bar{A} = 4\end{array} \xRightarrow{\text{Kron.-Kap.}}$ sistem nema rješenje

3° $\lambda \neq -2$ i $\lambda \neq 0$ $\text{rang}\, A = \text{rang}\, \bar{A} = 4 \xRightarrow{\text{Kron.-Kap.}}$ sistem ima tačno jedno rješenje

rješ. sist. $(x_1, x_2, x_3, x_4) = \left(1, 5, 3, \frac{1}{\lambda}\right)$

(#) Rješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra $\lambda$.

$$\begin{aligned} -x + 6y + (\lambda+3)z &= 21 \\ -x + 3y + 2z &= 9 \\ x + 3y + 2\lambda z &= \lambda + 13 \end{aligned}$$

Rj.-upute:

Sistem rješimo metodom determinanti (upotrebom Cramerovog pravila)

$$D = \begin{vmatrix} -1 & 6 & \lambda+3 \\ -1 & 3 & 2 \\ 1 & 3 & 2\lambda \end{vmatrix} = \dots = 0 \qquad D_x = \begin{vmatrix} 21 & 6 & \lambda+3 \\ 9 & 3 & 2 \\ \lambda+13 & 3 & 2\lambda \end{vmatrix} = \dots =$$

$$= (-3)(\lambda-1)(\lambda-2)$$

$$D_y = \begin{vmatrix} -1 & 21 & \lambda+3 \\ -1 & 9 & 2 \\ 1 & \lambda+13 & 2\lambda \end{vmatrix} = \dots = -(\lambda+1)(\lambda-2) \qquad D_z = \begin{vmatrix} -1 & 6 & 21 \\ -1 & 3 & 9 \\ 1 & 3 & \lambda+13 \end{vmatrix} = \dots = 3(\lambda-2)$$

Diskusija

1° $\lambda \neq 2 \Rightarrow D = 0$ i npr. $D_z \neq 0$

sistem nema rješenja

2° $\lambda = 2 \Rightarrow D = D_x = D_y = D_z = 0$ pa sistem moramo riješiti na drugi način

Za $\lambda=2$ sistem postaje

$$\begin{aligned} -x + 6y + 5z &= 21 \\ -x + 3y + 2z &= 9 \\ x + 3y + 4z &= 15 \end{aligned}$$

$$\left[\begin{array}{ccc|c} -1 & 6 & 5 & 21 \\ -1 & 3 & 2 & 9 \\ 1 & 3 & 4 & 15 \end{array}\right] \sim \dots \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3-t \\ 4-t \\ t \end{pmatrix},\ t \in R$$

rješenje sistema za $\lambda = 2$

(#) Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra λ.

$$\begin{aligned} -x + 8y + (\lambda+4)z &= 29 \\ -x + 4y + 3z &= 13 \\ x + 4y + (2\lambda-1)z &= \lambda+16 \end{aligned}$$

Rj.-upute:

Sistem rješimo upotrebom Cramerovog pravila (metodom determin.)

$$D = \begin{vmatrix} -1 & 8 & \lambda+4 \\ -1 & 4 & 3 \\ 1 & 4 & 2\lambda-1 \end{vmatrix} \underset{II_v+III_v}{\overset{I_v+III_v}{=\!=}} \begin{vmatrix} 0 & 12 & 3\lambda+3 \\ 0 & 8 & 2\lambda+2 \\ 1 & 4 & 2\lambda-1 \end{vmatrix} = \dots = 0$$

$$D_x = \begin{vmatrix} 29 & 8 & \lambda+ \\ 13 & 4 & 3 \\ \lambda+16 & 4 & 2\lambda-1 \end{vmatrix} = 4 \begin{vmatrix} 29 & 2 & \lambda+3 \\ 13 & 1 & 3 \\ \lambda+16 & 1 & 2\lambda-1 \end{vmatrix} = \dots = (-4)(\lambda-2)(\lambda-3)$$

$$D_y = \begin{vmatrix} -1 & 29 & \lambda+4 \\ -1 & 13 & 3 \\ 1 & \lambda+16 & 2\lambda-1 \end{vmatrix} = \dots = (-1)(\lambda+1)(\lambda-3), \quad D_z = \begin{vmatrix} -1 & 8 & 29 \\ -1 & 4 & 13 \\ 1 & 4 & \lambda+16 \end{vmatrix} = \dots = 4(\lambda-3)$$

Diskusija

1° $\lambda \neq 3 \Rightarrow D=0$ i npr. $D_z \neq 0$

sistem nema rješenje

2° $\overset{D=D_x=D_y=D_z=0}{\Uparrow}$ $\lambda = 3$ sistem postaje

$$\begin{aligned} -x + 8y + 7z &= 29 \\ -x + 4y + 3z &= 13 \\ x + 4y + 5z &= 21 \end{aligned}$$

Sistem npr. možemo riješiti Kron.-Kap metodom:

$$\left[\begin{array}{ccc|c} -1 & 8 & 7 & 29 \\ -1 & 4 & 3 & 13 \\ 1 & 4 & 5 & 19 \end{array}\right] \sim \dots \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rješenje sistema je:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3-s \\ 4-s \\ s \end{pmatrix}, \quad s \in \mathbb{R}$$

(#) Rješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra λ

$$
\begin{aligned}
-x + 10y + (\lambda+5)z &= 37\\
-x + 5y + 4z &= 17\\
x + 5y + (2\lambda-2)z &= \lambda+19
\end{aligned}
$$

Rj.-upute:

Sistem rješimo upotrebom Kramerovog pravila

$$D = \begin{vmatrix} -1 & 10 & \lambda+5 \\ -1 & 5 & 4 \\ 1 & 5 & 2\lambda-2 \end{vmatrix} = \ldots = 0 \qquad D_x = \begin{vmatrix} 37 & 10 & \lambda+5 \\ 17 & 5 & 4 \\ \lambda+19 & 5 & 2\lambda-2 \end{vmatrix} = \ldots = (-5)(\lambda-3)(\lambda-4)$$

$$D_y = \begin{vmatrix} -1 & 37 & \lambda+5 \\ -1 & 17 & 4 \\ 1 & \lambda+19 & 2\lambda-2 \end{vmatrix} = \ldots = (-1)(\lambda+1)(\lambda-4) \qquad D_z = \begin{vmatrix} -1 & 10 & 37 \\ -1 & 5 & 17 \\ 1 & 5 & \lambda+19 \end{vmatrix} = \ldots = 5(\lambda-4)$$

Diskusija

1° $\lambda \neq 4 \Rightarrow D=0,\ D_z \neq 0 \Rightarrow$ sistem nema rješenje

2° $\lambda = 4 \Rightarrow D = D_x = D_y = D_z = 0 \Rightarrow$ sistem rješimo na drugi način upr. Kroneker-Kapelijevom metodom

Sistem postaje

$$
\begin{aligned}
-x + 10y + 9z &= 37\\
-x + 5y + 4z &= 17\\
x + 5y + 6z &= 23
\end{aligned}
$$

$$\left[\begin{array}{ccc|c} -1 & 10 & 9 & 37 \\ -1 & 5 & 4 & 17 \\ 1 & 5 & 6 & 23 \end{array}\right] \sim \ldots \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3-s \\ 4-s \\ s \end{pmatrix},\ s \in \mathbb{R}$$

rješenje sistema

(#) Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra $\lambda$

$$\lambda x + 2y + z = 3$$
$$-9x - 2\lambda y + 3z = \lambda$$
$$8x + \lambda y + 2z = 6.$$

Rj-upute:

Sistem rješimo Cramerovom metodom (tj. metodom determinanti)

$$D = \begin{vmatrix} \lambda & 2 & 1 \\ -9 & -2\lambda & 3 \\ 8 & \lambda & 2 \end{vmatrix} \begin{matrix} II_v + I_v \cdot (-2) \\ III_v + I_v \cdot (-2) \end{matrix} \ldots = (-7)(\lambda+3)(\lambda-4)$$

$$D_x = \begin{vmatrix} 3 & 2 & 1 \\ \lambda & -2\lambda & 3 \\ 6 & \lambda & 2 \end{vmatrix} = \ldots = (\lambda-4)(\lambda-9) \qquad D_y = \begin{vmatrix} \lambda & 3 & 1 \\ -9 & \lambda & 3 \\ 8 & 6 & 2 \end{vmatrix} = \ldots = 2(\lambda-4)(\lambda-9$$

$$D_z = \begin{vmatrix} \lambda & 2 & 3 \\ -9 & -2\lambda & \lambda \\ 8 & \lambda & 6 \end{vmatrix} = -(\lambda-4)(\lambda^2+16\lambda+27)$$

Diskusija

1° $\lambda \neq 4$ i $\lambda \neq -3 \Rightarrow D \neq 0$ sistem ima jedinstveno rješenje

$$x = \frac{D_x}{D} = \frac{\lambda-9}{(-7)(\lambda+3)}; \quad y = \frac{D_y}{D} = \frac{2(\lambda-9)}{(-7)(\lambda+3)}; \quad z = \frac{D_z}{D} = \frac{-(\lambda^2+16\lambda+27)}{(-7)(\lambda+3)}$$

2° $\lambda = -3 \Rightarrow D = 0,\ D_x \neq 0$ sistem nema rješenja

3° $\lambda = 4 \Rightarrow D = 0,\ D_x = D_y = D_z = 0$ sistem rješimo na drugi način

$$\bar{A} = \left[\begin{array}{ccc|c} 4 & 2 & 1 & 3 \\ -9 & -8 & 3 & 4 \\ 8 & 4 & 2 & 6 \end{array}\right] \sim \ldots \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{16}{7} \\ 0 & 1 & -\frac{3}{2} & -\frac{43}{14} \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \operatorname{rang}(A) = \operatorname{rang}(\bar{A}) < 3$$

sistem ima $\infty$ mnogo rješenja

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{16}{7} - t \\ -\frac{43}{14} + \frac{3}{2}t \\ t \end{pmatrix}$$

$x = \frac{16}{7} - t$

$y = -\frac{43}{14} + \frac{3}{2}t \qquad t \in \mathbb{R}$

$z = t$

(#) Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra $\lambda$

$$\begin{aligned} 2x+(2\lambda-4)y+(\lambda-3)z &= 8 \\ 2x+(\lambda-2)y \qquad\qquad &= 5 \\ -3x \qquad\qquad +(\lambda-3)z &= -3 \end{aligned}$$

Rj.-upute:

Sistem rješimo Cramerovom metodom (tj. metodom determinanti)

$$D=\begin{vmatrix} 2 & 2\lambda-4 & \lambda-3 \\ 2 & \lambda-2 & 0 \\ -3 & 0 & \lambda-3 \end{vmatrix}=\ldots=(\lambda-2)(\lambda-3) \qquad D_x=\begin{vmatrix} 8 & 2\lambda-4 & \lambda-3 \\ 5 & \lambda-2 & 0 \\ -3 & 0 & \lambda-3 \end{vmatrix}=\ldots=(\lambda-2)(\lambda-3)$$

$$D_y=\begin{vmatrix} 2 & 8 & \lambda-3 \\ 2 & 5 & 0 \\ -3 & -3 & \lambda-3 \end{vmatrix}=\ldots=3(\lambda-3) \qquad D_z=\begin{vmatrix} 2 & 2\lambda-4 & 8 \\ 2 & \lambda-2 & 5 \\ -3 & 0 & -3 \end{vmatrix}=\ldots=0$$

Diskusija

1° $\lambda\neq 2$ i $\lambda\neq 3$ $\Rightarrow$ $D\neq 0$ sistem ima jedinstveno rješenje

$$x=1, \quad y=\frac{3}{\lambda-2}, \quad z=0$$

2° $\lambda=2$ $\Rightarrow$ $D=0$, $D_y\neq 0$ sistem nema rješenje

3° $\lambda=3$ $\Rightarrow$ $D=D_x=D_y=D_z=0$ $\Rightarrow$ sistem rješavamo na drugi način

$$\left[\begin{array}{ccc|c} 2 & 2 & 0 & 8 \\ 2 & 1 & 0 & 5 \\ -3 & 0 & 0 & -3 \end{array}\right] \sim \ldots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rješenje sistema u ovom slučaju je

$(1, 3, t)$, $t\in R$

tj. $x=1$, $y=3$, $z=t$, $t\in R$.

(#) Rješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra $\lambda$

$$\begin{aligned} x + 2y + \lambda z &= 1 \\ 2x + (\lambda+1)y + (2\lambda+2)z &= 2 \\ -3x - 6y + (4-2\lambda)z &= -6 \end{aligned}$$

Rj.-upute:

Sistem rješimo Cramerovom metodom

$$D = \begin{vmatrix} 1 & 2 & \lambda \\ 2 & \lambda+1 & 2\lambda+2 \\ -3 & -6 & 4-2\lambda \end{vmatrix} = \ldots = (\lambda+4)(\lambda-3) \qquad D_x = \begin{vmatrix} 1 & 2 & \lambda \\ 2 & \lambda+1 & 2\lambda+2 \\ -6 & -6 & 4-2\lambda \end{vmatrix} = \ldots = 4(\lambda^2-2\lambda-6)$$

$$D_y = \begin{vmatrix} 1 & 1 & \lambda \\ 2 & 2 & 2\lambda+2 \\ -3 & -6 & 4-2\lambda \end{vmatrix} = \ldots = 6 \qquad D_z = \begin{vmatrix} 1 & 2 & 1 \\ 2 & \lambda+1 & 2 \\ -3 & -6 & -6 \end{vmatrix} = (-3)(\lambda-3)$$

Diskusija

1° $\lambda \neq -4$ i $\lambda \neq 3$ $\Rightarrow$ $D \neq 0$ sistem ima jedinstveno rješenje.

$$x = \frac{D_x}{D} = \frac{4(\lambda^2-2\lambda-6)}{(\lambda+4)(\lambda-3)}; \quad y = \frac{D_y}{D} = \frac{6}{(\lambda+4)(\lambda-3)}; \quad z = \frac{D_z}{D} = \frac{(-3)}{\lambda+4}$$

2° $\lambda = -4$ $\Rightarrow$ $D=0$, $D_y \neq 0$ sistem nema rješenje.

3° $\lambda = 3$ $\Rightarrow$ $D=0$, $D_y \neq 0$ sistem nema rješenje.

(#) Riješiti sistem jednačina

$$\begin{aligned} 2x_1 + 5x_2 - 8x_3 &= 8 \\ 4x_1 + 3x_2 - 9x_3 &= 9 \\ 2x_1 + 3x_2 - 5x_3 &= 7 \\ x_1 + 8x_2 - 7x_3 &= 12 . \end{aligned}$$

Rj.-upute

Sistem rješimo Kroneker-Kapelijevom metodom

$$\bar{A} = \left[\begin{array}{ccc|c} 2 & 5 & -8 & 8 \\ 4 & 3 & -9 & 9 \\ 2 & 3 & -5 & 7 \\ 1 & 8 & -7 & 12 \end{array}\right] \sim \ldots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Sistem ima tačno jedno rješenje

$x_1 = 3$

$x_2 = 2$

$x_3 = 1$

(#) Riješiti sistem i diskutovati rješenja sistema u zavisnosti od parametra $\lambda$

$$\begin{aligned} x_1 \phantom{+(2-\lambda)x_2} &+ x_3 + x_4 = 1 \\ 2x_1 + (2-\lambda)x_2 &+ 3x_3 + 3x_4 = 7-\lambda \\ x_1 + (2-\lambda)x_2 &+ x_3 + x_4 = 3-\lambda \end{aligned}$$

Rj.-upute:

Sistem riješimo Kroneker-Kapelijevom metodom

$$\bar{A} = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 2 & 2-\lambda & 3 & 3 & 7-\lambda \\ 1 & 2-\lambda & 1 & 1 & 3-\lambda \end{array}\right] \overset{II_k \leftrightarrow IV_k}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 2 & 3 & 3 & 2-\lambda & 7-\lambda \\ 1 & 1 & 1 & 2-\lambda & 3-\lambda \end{array}\right] \sim$$

$$\sim \ldots \sim \begin{array}{c} \begin{array}{cccc} x_1 & x_4 & x_3 & x_2 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 2-\lambda & 2-\lambda \end{array}\right] \end{array} \quad \ldots (*)$$

Diskusija

$1°$ $\lambda = 2$

$$\left.\begin{array}{l} \text{rang}(A) = 2 \\ \text{rang}(\bar{A}) = 2 \\ \text{broj nepoznatih} = 4 \end{array}\right\} \xrightarrow{\text{Kron.-Kap.}}$$

sistem ima $\infty$ mnogo rješenja i dvije promjenjive uzimamo proizvoljno npr prema (*) možemo uzeti

$x_3 = t,\ t \in \mathbb{R},\quad x_3 = s,\ s \in \mathbb{R}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ s \\ t \\ 3-t \end{pmatrix},\ s,t \in \mathbb{R}$$

$2°$ $\lambda \neq 2$

$$\left.\begin{array}{l} \text{rang}(A) = 3 \\ \text{rang}(\bar{A}) = 3 \\ \text{broj nepoznatih} = 4 \end{array}\right\} \xrightarrow{\text{Kron.-Kap.}}$$

sistem ima $\infty$ mnogo rješenja i jednu promjenjivu uzimamo proizvoljno npr. iz (*) $x_4 = t$

$x_1 = -2,\ x_2 = 1,\ x_3 = 3-t,\ x_4 = t$

(#) Riješiti sistem i diskutovati rješenja sistema u zavisnosti od parametra $\lambda$

$$\begin{aligned} x_1 + x_2 + x_3 \phantom{+ (\lambda-1)x_4} &= -1 \\ 2x_1 + 3x_2 + 3x_3 + (\lambda-1)x_4 &= -1 \\ 3x_1 + 4x_2 + 4x_3 + (2\lambda-2)x_4 &= 2 \end{aligned}$$

Rj.-upute

Sistem rješimo Kroneker-Kapelijevom metodom

$$\bar{A} = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -1 \\ 2 & 3 & 3 & \lambda-1 & -1 \\ 3 & 4 & 4 & 2\lambda-2 & 2 \end{array}\right] \sim \dots \sim \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & \lambda-1 & 4 \end{array}\right] \end{array}$$

$\dots (*)$

Diskusija:

1° $\lambda = 1$

$$\left.\begin{array}{l} \operatorname{rang}(A) = 2 \\ \operatorname{rang}(\bar{A}) = 3 \end{array}\right\} \overset{\text{Kron.-Kap.}}{\Longrightarrow}$$ sistem nema rješenja

2° $\lambda \neq 1$

$$\left.\begin{array}{l} \operatorname{rang}(A) = 3 \\ \operatorname{rang}(\bar{A}) = 3 \\ \text{broj nepoznatih} = 4 \end{array}\right\} \overset{\text{Kron.-Kap.}}{\Longrightarrow}$$ sistem ima $\infty$ mnogo rješenja i jednu promjenjivu uzimamo proizvoljno

Prema $(*)$ kao proizvoljnu promjenjivu uzet ćemo $x_3$, tj. $x_3 = t$, $t \in R$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ -3-t \\ t \\ \frac{4}{\lambda-1} \end{pmatrix}, \quad t \in R$$

traženo rješenje

(#) Dat je skup $\mathcal{B}=\left\{\begin{pmatrix}3\\-6\\-9\end{pmatrix},\begin{pmatrix}2\\-5\\-6\end{pmatrix},\begin{pmatrix}-1\\1\\5\end{pmatrix}\right\}$. Provjeriti da li je skup $\mathcal{B}$ linearno nezavisan. Da li je $\mathcal{B}$ baza vektorskog prostora $\mathbb{R}^3$. Zašto? Vektor $u=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ izraziti kao linearnu kombinaciju vektora iz baze $\mathcal{B}$ (drugim riječima odrediti koordinate vektora $u=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ u odnosu na bazu $\mathcal{B}$).

Rj.-upute:

Skup $\mathcal{B}$ je linearno nezavisan akko jedino rješenje sistema

$$\alpha\begin{pmatrix}3\\-6\\-9\end{pmatrix}+\beta\begin{pmatrix}2\\-5\\-6\end{pmatrix}+\gamma\begin{pmatrix}-1\\1\\5\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$ po nepoznatim $\alpha, \beta$ i $\gamma$

je trivijalno rješenje $\alpha=\beta=\gamma=0$.

$$\begin{aligned}3\alpha+2\beta-\gamma&=0\\-6\alpha+(-5)\beta+\gamma&=0\\-9\alpha-6\beta+5\gamma&=0\end{aligned}$$

ovo je homogeni sistem (uvijek ima jedno rješenje)

$$D=\begin{vmatrix}3&2&-1\\-6&-5&1\\-9&-6&5\end{vmatrix}=-6\neq 0$$

$D\neq 0$ skup $\mathcal{B}$ je linearno nezavisan

$\mathcal{B}$ jest baza vektorskog prostora $\mathbb{R}^3$ zato što skup bilo koja tri linearno nezavisna vektora formiraju bazu od $\mathbb{R}^3$.

Koordinate vektora $u$ u odnosu na bazu $\mathcal{B}$ su $\begin{pmatrix}2\\-2\\1\end{pmatrix}$, drugim riječima

$$u=2\begin{pmatrix}3\\-6\\-9\end{pmatrix}-2\begin{pmatrix}2\\-5\\-6\end{pmatrix}+\begin{pmatrix}-1\\1\\5\end{pmatrix}.$$

(#) Dat je skup $\mathcal{B}=\left\{\begin{pmatrix}3\\-9\\6\end{pmatrix},\begin{pmatrix}2\\-7\\4\end{pmatrix},\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}$. Provjeriti da li je skup $\mathcal{B}$ linearno nezavisan. Objasniti zašto je $\mathcal{B}$ baza vektorskog prostora $\mathbb{R}^3$. Vektor $u=\begin{pmatrix}3\\-4\\4\end{pmatrix}$ izraziti kao linearnu kombinaciju vektora iz baze $\mathcal{B}$ (drugim riječima odrediti koordinate vektora $u=\begin{pmatrix}3\\-4\\4\end{pmatrix}$ u odnosu na bazu $\mathcal{B}$).

Rj.-upute:

Skup $\mathcal{B}$ je linearno nezavisan akko jedino rješenje sistema

$$\alpha\begin{pmatrix}3\\-9\\6\end{pmatrix}+\beta\begin{pmatrix}2\\-7\\4\end{pmatrix}+\gamma\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

po nepoznatim $\alpha, \beta$ i $\gamma$, je trivijalno rješenje $\alpha=\beta=\gamma=0$.

$$\begin{aligned}3\alpha+2\beta+\gamma&=0\\-9\alpha-7\beta-\gamma&=0\\6\alpha+4\beta+\gamma&=0\end{aligned}$$

homogeni sistem

$$D=\begin{vmatrix}3&2&1\\-9&-7&-1\\6&4&1\end{vmatrix}=3$$

Skup od Bilo koja tri linearno nezavisna vektora formiju bazu za $\mathbb{R}^3$ pa je $\mathcal{B}$ baza vektorskog prostora $\mathbb{R}^3$.

Koordinate vektora $u=\begin{pmatrix}3\\-4\\4\end{pmatrix}$ u odnosu na bazu $\mathcal{B}$ su $\begin{pmatrix}1\\-1\\2\end{pmatrix}$, drugim riječima

$$u=\begin{pmatrix}3\\-9\\6\end{pmatrix}-\begin{pmatrix}2\\-7\\4\end{pmatrix}+2\begin{pmatrix}1\\-1\\1\end{pmatrix}.$$

(#) Vektor $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} \right\}$ ima koordinate $\begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix}$. Odrediti koordinate vektora $v$ u odnosu na bazu $\mathcal{B}' = \left\{ \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$.

Rj.-upute.

Posmatrajmo baze $\mathcal{B}$ i $\mathcal{B}'$. Nije teško vidjeti da je

$$\left.\begin{aligned}
\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} &= 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} &= 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} &= (-1) \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}
\end{aligned}\right\} \ldots (*)$$

Kako su koordinate vektora $v$ u odnosu na bazu $\mathcal{B}$ $\begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix}$ to znači da je $v = 4 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} + 7 \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix}$.

Prema $(*)$ imamo

$$4 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} = 4 \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \qquad + 4 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$(-1) \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} = (-1) \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + (-1) \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

$$7 \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} = (-7) \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

Prema tome $v = (-4) \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Koordinate vektora $v$ u odnosu na bazu $\mathcal{B}'$ su $\begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix}$.

(#) Vektor $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}$ ima koordinate $\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$. Odrediti koordinate vektora $v$ u odnosu na bazu $\mathcal{B}' = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Rj.-upute:

Posmatrajmo baze $\mathcal{B}$ i $\mathcal{B}'$. Nije teško vidjeti da je

$$\left. \begin{aligned} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} &= 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} &= (-1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} &= 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned} \right\} \quad \ldots (*)$$

Kako su koordinate vektora $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B}$ $\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$ to znači da je $v = 5 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$.

Prema $(*)$ imamo

$$\begin{aligned} 5 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} &= 5 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ (-1) \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} &= 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ 3 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} &= 6 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

Prema tome $v = 6 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 9 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Koordinate vektora $v$ u odnosu na bazu $\mathcal{B}'$ su $\begin{pmatrix} 6 \\ 4 \\ 9 \end{pmatrix}$.

(#) Vektor $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ ima koordinate $\begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$. Odrediti koordinate vektora $v$ u odnosu na bazu $\mathcal{B}' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Rj.-upute:

Posmatrajmo baze $\mathcal{B}$ i $\mathcal{B}'$. Nije teško vidjeti da je

$$\left.\begin{aligned}
\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} &= 0 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} &= (-1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} &= 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\end{aligned}\right\} \ldots (*)$$

Kako su koordinate vektora $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B}$ $\begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$ to znači da je $v = 7 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Sad prema $(*)$ imamo

$$\begin{aligned}
7 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} &= 7 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\
-3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} &= 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-3) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
5 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} &= 5 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-5) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\end{aligned}$$

Prema tome $v = 8 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Koordinate vektora $v$ u odnosu na bazu $\mathcal{B}'$ su $\begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix}$.

\# Vektor $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}$ ima koordinate $\begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix}$. Odrediti koordinate vektora $v$ u odnosu na bazu $\mathcal{B}' = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Rj.-upute:

Posmatrajmo baze $\mathcal{B}$ i $\mathcal{B}'$. Nije teško vidjeti da je

$$\left. \begin{aligned} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} &= 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} &= 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} &= (-1) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{aligned} \right\} \quad \ldots (*)$$

Kako su koordinate vektora $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B}$ $\begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix}$ to znači da je $v = 6 \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$.

Prema $(*)$ imamo

$$\begin{aligned} 6 \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} &= 6 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ -2 \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} &= (-2) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + (-4) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ 4 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} &= (-4) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

Prema tome $v = 0 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 12 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Koordinate vektora $v$ u odnosu na bazu $\mathcal{B}'$ su $\begin{pmatrix} 0 \\ 12 \\ 6 \end{pmatrix}$.

(#) Odrediti sve vrijednosti parametra m tako da vektori $\vec{a}=\begin{pmatrix} m-2\\ 1\\ 2\end{pmatrix}$, $\vec{b}=\begin{pmatrix} m-2\\ m-2\\ 3\end{pmatrix}$, $\vec{c}=\begin{pmatrix} m-2\\ 1\\ m-2\end{pmatrix}$ nisu baza (ne čine bazu) vektorskog prostora $\mathbb{R}^3$. Za najveću dobijenu vrijednost parametra m izraziti vektor $\vec{c}$ kao linearnu kombinaciju vektora $\vec{a}$ i $\vec{b}$.

Rj.-uputa.

Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ neće činiti bazu vektorskog prostora $\mathbb{R}^3$ ako su linearno zavisni, a oni su linearno zavisni ako postoje brojevi $\alpha$, $\beta$ i $\gamma$ (ne svi nula) taki da $\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=0$,

$$\alpha\begin{pmatrix} m-2\\ 1\\ 2\end{pmatrix}+\beta\begin{pmatrix} m-2\\ m-2\\ 3\end{pmatrix}+\gamma\begin{pmatrix} m-2\\ 1\\ m-2\end{pmatrix}=0 \iff \underbrace{\begin{pmatrix} m-2 & m-2 & m-2\\ 1 & m-2 & 1\\ 2 & 3 & m-2\end{pmatrix}}_{=M}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

a ovaj sistem će imati netrivijalna rješenja za

$\det M \neq 0$

$$\det M=\begin{vmatrix} m-2 & m-2 & m-2\\ 1 & m-2 & 1\\ 2 & 3 & m-2\end{vmatrix}=(m-2)\begin{vmatrix} 1 & 1 & 1\\ 1 & m-2 & 1\\ 2 & 3 & m-2\end{vmatrix}=\ldots=(m-2)(m-3)(m-4)$$

Za $m\in\{2,3,4\}$ dati vektori nisu baza prostora $\mathbb{R}^3$.

Za $m=4$ imamo $\vec{a}=\begin{pmatrix}2\\ 1\\ 2\end{pmatrix}$, $\vec{b}=\begin{pmatrix}2\\ 2\\ 3\end{pmatrix}$, $\vec{c}=\begin{pmatrix}2\\ 1\\ 2\end{pmatrix}$

$$\vec{c}=\eta\begin{pmatrix}2\\ 1\\ 2\end{pmatrix}+\mu\begin{pmatrix}2\\ 2\\ 3\end{pmatrix} \Rightarrow \begin{matrix}\eta=1\\ \mu=0\end{matrix} \qquad \vec{c}=\vec{a}+0\cdot\vec{b}.$$

(#) Odrediti sve vrijednosti parametra m tako da vektori $\vec{a}=\begin{pmatrix} m-1\\ m-1\\ m-1\end{pmatrix}$, $\vec{b}=\begin{pmatrix} 1\\ m-1\\ 1\end{pmatrix}$, $\vec{c}=\begin{pmatrix} 2\\ 3\\ m-1\end{pmatrix}$ nisu baza (ne čine bazu) vektorskog prostora $\mathbb{R}^3$. Za najveću dobijenu vrijednost parametra m izraziti vektor $\vec{c}$ kao linearnu kombinaciju vektora $\vec{a}$ i $\vec{b}$.

Rj.-uputa:

Za vektore $\vec{a}$, $\vec{b}$ i $\vec{c}$ kažemo da su linearno zavisni ako postoje konstante $\alpha$, $\beta$ i $\gamma$ (ne sve jednake nuli) t.d. $\alpha\vec{a}\cancel{\beta\vec{b}}+\gamma\vec{c}=\vec{0}$

$$\alpha\begin{pmatrix} m-1\\ m-1\\ m-1\end{pmatrix}+\beta\begin{pmatrix} 1\\ m-1\\ 1\end{pmatrix}+\gamma\begin{pmatrix} 2\\ 3\\ m-1\end{pmatrix}=0 \iff \underbrace{\begin{bmatrix} m-1 & 1 & 2\\ m-1 & m-1 & 3\\ m+1 & 1 & m-1\end{bmatrix}}_{M}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Sistem će imati jedinstveno rješenje kada je $\det M=0$ tj. za $\det M\neq 0$ vektori su linearno zavisni i neće formirati bazu za $\mathbb{R}^3$.

$$\det M=\begin{vmatrix} m-1 & 1 & 2\\ m-1 & m-1 & 3\\ m-1 & 1 & m-1\end{vmatrix}=(m-1)\begin{vmatrix} 1 & 1 & 2\\ 1 & m-1 & 3\\ 1 & 1 & m-1\end{vmatrix}=\ldots=(m-1)(m-2)(m-3)$$

Za $m\in\{1,2,3\}$ dati vektori nisu baza prostora $\mathbb{R}^3$.

Za $m=3$ imamo $\vec{a}=\begin{pmatrix}2\\2\\2\end{pmatrix}$, $\vec{b}=\begin{pmatrix}1\\2\\1\end{pmatrix}$, $\vec{c}=\begin{pmatrix}2\\3\\2\end{pmatrix}$

$$\vec{c}=\eta\begin{pmatrix}2\\2\\2\end{pmatrix}+\mu\begin{pmatrix}1\\2\\1\end{pmatrix} \Rightarrow \begin{matrix}\eta=\frac{1}{2}\\ \mu=1\end{matrix} \quad \text{tj. } \vec{c}=\frac{1}{2}\vec{a}+\vec{b}$$

(#) Ako je $\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$ jedna baza vektorskog prostora $\mathbb{R}^3$, dokazati da i vektori $\vec{b_1} = \vec{a_2} + 3\vec{a_3}$, $\vec{b_2} = \vec{a_1} + \vec{a_2} + 2\vec{a_3}$ i $\vec{b_3} = 2\vec{a_1} + 2\vec{a_2} + 6\vec{a_3}$ također čine baza prostora $\mathbb{R}^3$ i izraziti vektor $\vec{c} = -\vec{a_1} + \vec{a_2} + 2\vec{a_3}$ preko vektora baze $\{\vec{b_1}, \vec{b_2}, \vec{b_3}\}$.

Rj.-upute:

Vektori $\vec{b_1}, \vec{b_2}$ i $\vec{b_3}$ će činiti bazu prostora $\mathbb{R}^3$ ako su linearno nezavisni tj. ako je jedino rješenje sistema

$$\alpha \vec{b_1} + \beta \vec{b_2} + \gamma \vec{b_3} = \vec{0}$$

trivijalno rješenje $\alpha = \beta = \gamma = 0$. Posmatrajmo dati sistem

$$\alpha(\vec{a_2} + 3\vec{a_3}) + \beta(\vec{a_1} + \vec{a_2} + 2\vec{a_3}) + \gamma(2\vec{a_1} + 2\vec{a_2} + 6\vec{a_3}) = \vec{0}$$

$$(0 + \beta + 2\gamma)\vec{a_1} + (\alpha + \beta + 2\gamma)\vec{a_2} + (3\alpha + 2\beta + 6\gamma)\vec{a_3} = \vec{0}$$

$$\begin{aligned} \beta + 2\gamma &= 0 \\ \alpha + \beta + 2\gamma &= 0 \\ 3\alpha + 2\beta + 6\gamma &= 0 \end{aligned} \iff \underbrace{\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 3 & 2 & 6 \end{pmatrix}}_{=M} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\det M = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 3 & 2 & 6 \end{vmatrix} = \ldots = -2 \neq 0 \Rightarrow$$ vektori $\vec{b_1}, \vec{b_2}$ i $\vec{b_3}$ su linearno nezavisni i oni čine bazu prostora $\mathbb{R}^3$

Odredimo još konstante $c_1, c_2$ i $c_3$ t.d. $\vec{c} = c_1\vec{b_1} + c_2\vec{b_2} + c_3\vec{b_3}$

$$-\vec{a_1} + \vec{a_2} + 2\vec{a_3} = c_1(a_2 + 3\vec{a_3}) + c_2(\vec{a_1} + \vec{a_2} + 2\vec{a_3}) + c_3(2\vec{a_1} + 2\vec{a_2} + 6\vec{a_3})$$

$$\begin{aligned} c_2 + 2c_3 &= -1 \\ c_1 + c_2 + 2c_3 &= 1 \\ 3c_1 + 2c_2 + 6c_3 &= 2 \end{aligned} \iff \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 3 & 2 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{aligned} c_1 &= 2 \\ c_2 &= 1 \\ c_3 &= -1 \end{aligned}$$

(#) Ako je $\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$ jedna baza vektorskog prostora $\mathbb{R}^3$, dokazati da i vektori $\vec{b_1} = \vec{a_1} + 2\vec{a_2} + 3\vec{a_3}$, $\vec{b_2} = \vec{a_1} + \vec{a_2} + 2\vec{a_3}$ i $\vec{b_3} = 2\vec{a_1} + \vec{a_2} + 4\vec{a_3}$ također čine bazu prostora $\mathbb{R}^3$ i izraziti vektor $\vec{c} = \vec{a_1} + \vec{a_2} + \vec{a_3}$ preko vektora baze $\{\vec{b_1}, \vec{b_2}, \vec{b_3}\}$.

Rješenje-upute:

Vektori $\vec{b_1}$, $\vec{b_2}$ i $\vec{b_3}$ će činiti bazu prostora $\mathbb{R}^3$ ako su linearno nezavisni. Drugim riječima ako je jedino rješenje sistema $\alpha\vec{b_1} + \beta\vec{b_2} + \gamma\vec{b_3} = \vec{0}$

trivijalno rješenje $\alpha = \beta = \gamma = 0$. Pa posmatrajmo dati sistem

$$\alpha\underbrace{(\vec{a_1} + 2\vec{a_2} + 3\vec{a_3})}_{=\vec{b_1}} + \beta\underbrace{(\vec{a_1} + \vec{a_2} + 2\vec{a_3})}_{=\vec{b_2}} + \gamma\underbrace{(2\vec{a_1} + \vec{a_2} + 4\vec{a_3})}_{=\vec{b_3}} = \vec{0}$$

$$(\alpha + \beta + 2\gamma)\vec{a_1} + (2\alpha + \beta + \gamma)\vec{a_2} + (3\alpha + 2\beta + 4\gamma)\vec{a_3} = \vec{0}$$

$$\begin{array}{l} \alpha + \beta + 2\gamma = 0 \\ 2\alpha + \beta + \gamma = 0 \\ 3\alpha + 2\beta + 4\gamma = 0 \end{array} \iff \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix}}_{=M} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\det M = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{vmatrix} = \ldots = -1 \neq 0 \Rightarrow$$ vektori $\vec{b_1}$, $\vec{b_2}$ i $\vec{b_3}$ su linearno nezavisni

Odredimo konstante $c_1$, $c_2$ i $c_3$ t.d. $\vec{c} = c_1\vec{b_1} + c_2\vec{b_2} + c_3\vec{b_3}$

$$\vec{a_1} + \vec{a_2} + \vec{a_3} = c_1(\vec{a_1} + 2\vec{a_2} + 3\vec{a_3}) + c_2(\vec{a_1} + \vec{a_2} + 2\vec{a_3}) + c_3(2\vec{a_1} + \vec{a_2} + 4\vec{a_3})$$

$$\begin{array}{l} c_1 + c_2 + 2c_3 = 1 \\ 2c_1 + c_2 + c_3 = 1 \\ 3c_1 + 2c_2 + 4c_3 = 1 \end{array} \iff \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow c_1 = -1,\ c_2 = 4,\ c_3 = -1$$

(#) Za koje vrijednosti parametra $m$ vektori $\vec{a}=(2m,1+m,1)^T$, $\vec{b}=(-m,1,m)^T$, $\vec{c}=(m,1,m-2)^T$ čine bazu trodimenzionalnog vektorskog prostora?

Rj: Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ će činiti bazu trodimenzionalnog vektorskog prostora akko su linearno nezavisni tj. akko jedino rješenje sistema po nepoznatim $\alpha$ $\beta$ i $\gamma$

$$\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\vec{0}$$

je trivijalno rješenje $\alpha=\beta=\gamma=0$. Drugim riječima akko je determinanta $\begin{vmatrix} 2m & -m & m \\ 1+m & 1 & 1 \\ 1 & m & m-2 \end{vmatrix}$ različita od nule.

Pa izračunajmo vrijednost ove determinante.

$$\begin{vmatrix} 2m & -m & m \\ 1+m & 1 & 1 \\ 1 & m & m-2 \end{vmatrix} = m\begin{vmatrix} 2 & -1 & 1 \\ 1+m & 1 & 1 \\ 1 & m & m-2 \end{vmatrix} \overset{I_k+II_k\cdot 2}{\underset{III_k+II_k}{=}} m\begin{vmatrix} 0 & -1 & 0 \\ 3+m & 1 & 2 \\ 2m+1 & m & 2m-2 \end{vmatrix} =$$

$$= m\begin{vmatrix} 3+m & 2 \\ 2m+1 & 2m-2 \end{vmatrix} \overset{II_v+I_v}{=} m\begin{vmatrix} 3+m & 2 \\ 3m+4 & 2m \end{vmatrix} = m\left(6m+2m^2-6m-8\right)$$

$$= m(2m^2-8) = 2m(m-2)(m+2)$$

Za $m\neq 0$, $m\neq 2$, $m\neq -2$ vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ čine bazu trodimenzionalnog vektorskog prostora.

(#) Za koju vrijednost parametra $m$ vektori $\vec{a}=(m,-m,1)^T$, $\vec{b}=(-m,m,2m+2)^T$, $\vec{c}=(m,m+1,1-m)^T$ čine bazu trodimenzionalnog vektorskog prostora?

Rj.: Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ čine bazu trodimenzionalnog vektorskog prostora akko su linearno nezavisni, tj. akko jedino rješenje sistema

$$\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}$$

po nepoznatim $\alpha$, $\beta$ i $\gamma$ je trivijalno rješenje $\alpha=\beta=\gamma=0$. Drugim riječima akko je sljedeća determinanta različita od nule

$$\begin{vmatrix} m & -m & m \\ -m & m & m+1 \\ 1 & 2m+2 & 1-m \end{vmatrix}.$$

Izračunajmo ovu determinantu:

$$\begin{vmatrix} m & -m & m \\ -m & m & m+1 \\ 1 & 2m+2 & 1-m \end{vmatrix} = m\begin{vmatrix} 1 & -1 & 1 \\ -m & m & m+1 \\ 1 & 2m+2 & 1-m \end{vmatrix} \xlongequal[III_k+II_k]{I_k+II_k} m\begin{vmatrix} 0 & -1 & 0 \\ 0 & m & 2m+1 \\ 2m+3 & 2m+2 & m+3 \end{vmatrix}$$

$$= m\begin{vmatrix} 0 & 2m+1 \\ 2m+3 & m+3 \end{vmatrix} = -m(2m+1)(2m+3)$$

Za $m\neq 0$, $m\neq-\frac{1}{2}$, $m\neq-\frac{3}{2}$ vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ čine bazu trodimenzionalnog vektorskog prostora.

(#) Za koje vrijednosti parametra $m$ vektori $\vec{a}=(2m,1-m,1)^T$, $\vec{b}=(-2m,m,2m+2)^T$, $\vec{c}=(m,1+m,1-m)$ čine bazu trodimenzionalnog vektorskog prostora?

Rj. Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ će činiti bazu trodimenzionalnog vektorskog prostora akko su linearno nezavisni tj. akko jedino rješenje sistema

$$\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\vec{0}$$

(po nepoznatim $\alpha$, $\beta$ i $\gamma$) je trivijalno rješenje $\alpha=\beta=\gamma=0$. Drugim riječima akko je determinanta

$$\begin{vmatrix} 2m & -2m & 1 \\ 1-m & m & 2m+2 \\ 1 & 1+m & 1-m \end{vmatrix}$$

različita od nule.

Pa izračunajmo vrijednost date determinante.

$$\begin{vmatrix} 2m & -2m & m \\ 1-m & m & 1+m \\ 1 & 2m+2 & 1-m \end{vmatrix} = m\begin{vmatrix} 2 & -2 & 1 \\ 1-m & m & 1+m \\ 1 & 2m+2 & 1-m \end{vmatrix} \xrightarrow[II_k+III_k\cdot 2]{I_k+III_k\cdot(-2)} m\begin{vmatrix} 0 & 0 & 1 \\ -1-3m & 2+3m & 1+m \\ -1+2m & 4 & 1-m \end{vmatrix}$$

$$= m\begin{vmatrix} -1-3m & 2+3m \\ -1+2m & 4 \end{vmatrix} \xrightarrow{II_k+I_k} m\begin{vmatrix} -1-3m & 1 \\ -1+2m & 3+2m \end{vmatrix} =$$

$$= m\left(-3\underline{-2m}\,\underline{-9m}-6m^2+1\underline{-2m}\right) = m(-1)(6m^2+13m+2) =$$

$$= -m(m+2)(6m+1)$$

Za $m\neq 0$, $m\neq -2$, $m\neq -\frac{1}{6}$ vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ čine bazu trodimenzionalnog vektorskog prostora.

(#) Neka je $\mathcal{B}=\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ jedna baza prostora $\mathbb{R}^3$.

(a) Dokazati da je skup $\mathcal{B}'=\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ također baza prostora $\mathbb{R}^3$, gdje su $\vec{b}_1 = 14\vec{a}_1 - \vec{a}_2 + 32\vec{a}_3$, $\vec{b}_2 = 16\vec{a}_1 - \vec{a}_2 + 36\vec{a}_3$ i $\vec{b}_3 = -41\vec{a}_1 + 3\vec{a}_2 - 93\vec{a}_3$.

(b) Odredite koordinate vektora $\vec{a}_2$ u odnosu na bazu $\mathcal{B}'$.

Rj.-upute:

(a) Da bi $\mathcal{B}'$ bio baza prostora potrebno je i dovoljno da je on linearno nezavisan skup

$$\alpha\vec{b}_1 + \beta\vec{b}_2 + \gamma\vec{b}_3 = \vec{0}$$

$$\alpha(14\vec{a}_1 - \vec{a}_2 + 32\vec{a}_3) + \beta(16\vec{a}_1 - \vec{a}_2 + 36\vec{a}_3) + \gamma(-41\vec{a}_1 + 3\vec{a}_2 - 93\vec{a}_3) = \vec{0}$$

$$\begin{aligned} 14\alpha + 16\beta - 41\gamma &= 0 \\ -\alpha - \beta + 3\gamma &= 0 \\ 32\alpha + 36\beta - 93\gamma &= 0 \end{aligned} \qquad D = \begin{vmatrix} 14 & 16 & -41 \\ -1 & -1 & 3 \\ 32 & 36 & -93 \end{vmatrix} = \ldots = 2 \neq 0$$

$\Downarrow$
dati sistem ima
jedinstveno rješenje

$\Rightarrow$ skup $\mathcal{B}'$ je linearno nezavisan $\Rightarrow$ $\mathcal{B}'$ je također baza prostora $\mathbb{R}^3$

(b) Trebamo odrediti brojeve $\alpha, \beta$ i $\gamma$ takve da

$$\alpha\vec{b}_1 + \beta\vec{b}_2 + \gamma\vec{b}_3 = \vec{a}_2$$

Ovo se svodi na sistem

$$\begin{aligned} 14\alpha + 16\beta - 41\gamma &= 0 \\ -\alpha - \beta + 3\gamma &= 1 \\ 32\alpha + 36\beta - 93\gamma &= 0 \end{aligned}$$

Ovaj sistem možemo riješiti npr. Kroneker-Kapelijevom metodom

$$\left[\begin{array}{ccc|c} 14 & 16 & -41 & 0 \\ -1 & -1 & 3 & 1 \\ 32 & 36 & -93 & 0 \end{array}\right] \sim \ldots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Koordinate vektora $\vec{a}_2$ u odnosu na bazu $\mathcal{B}'$ su $\begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$.

(#) Neka je $\mathcal{B}=\{\vec{a_1},\vec{a_2},\vec{a_3}\}$ jedna baza prostora $\mathbb{R}^3$.

(a) Dokazati da je skup $\mathcal{B}'=\{\vec{b_1},\vec{b_2},\vec{b_3}\}$ također baza prostora $\mathbb{R}^3$, gdje su $\vec{b_1}=22\vec{a_1}+\vec{a_2}+39\vec{a_3}$, $\vec{b_2}=-24\vec{a_1}-\vec{a_2}-43\vec{a_3}$ i $\vec{b_3}=-2\vec{a_1}-3\vec{a_3}$.

(b) Odrediti koordinate vektora $\vec{a_2}$ u odnosu na bazu $\mathcal{B}'$ (drugim riječima izraziti vektor $\vec{a_2}$ preko vektora iz baze $\mathcal{B}'$)

Rj. -upute:

(a) Da bi skup $\mathcal{B}'$ bio baza potrebno je i dovoljno da je on linearno nezavisan skup

$$\alpha\vec{b_1}+\beta\vec{b_2}+\gamma\vec{b_3}=\vec{0}$$

$$\alpha(22\vec{a_1}+\vec{a_2}+39\vec{a_3})+\beta(-24\vec{a_1}-\vec{a_2}-43\vec{a_3})+\gamma(-2\vec{a_1}-3\vec{a_3})=\vec{0}$$

$$\begin{aligned} 22\alpha-24\beta-2\gamma&=0\\ \alpha-\beta&=0\\ 39\alpha-43\beta-3\gamma&=0\end{aligned} \qquad D=\begin{vmatrix}22&-24&-2\\1&-1&0\\39&-43&-3\end{vmatrix}=\ldots=2\neq 0$$

$\Downarrow$ dati sistem ima jedinstveno rješenje

$\Rightarrow$ skup $\mathcal{B}'$ je linearno nezavisan $\Rightarrow$ $\mathcal{B}'$ je također baza prostora $\mathbb{R}^3$

(b) Trebamo odrediti nepoznate skalare $\alpha,\beta$ i $\gamma$ takve da

$$\alpha\vec{b_1}+\beta\vec{b_2}+\gamma\vec{b_3}=\vec{a_2}$$

Na osnovu dijela pod (a) nije teško vidjeti da se ovo svodi na sistem

$$\begin{aligned} 22\alpha-24\beta-2\gamma&=0\\ \alpha-\beta&=1\\ 39\alpha-43\beta-3\gamma&=0\end{aligned}$$

Sistem možemo riješiti npr. Kronecker-Kapelijevom metodom:

$$\left[\begin{array}{ccc|c}22&-24&-2&0\\1&-1&0&1\\39&-43&-3&0\end{array}\right]\sim\ldots\sim\left[\begin{array}{ccc|c}1&0&0&7\\0&1&0&6\\0&0&1&5\end{array}\right]$$

Koordinate vektora $\vec{a_2}$ u odnosu na bazu $\mathcal{B}'$ su $\begin{pmatrix}7\\6\\5\end{pmatrix}$.

(#) Neka je $\mathcal{B}=\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$ jedna baza prostora $\mathbb{R}^3$.

(a) Dokazati da je skup $\mathcal{B}'=\{\vec{b_1}, \vec{b_2}, \vec{b_3}\}$ također baza prostora $\mathbb{R}^3$, gdje su $\vec{b_1}=15\vec{a_1}-\vec{a_2}+33\vec{a_3}$, $\vec{b_2}=3\vec{a_1}+6\vec{a_3}$ i $\vec{b_3}=-29\vec{a_1}+2\vec{a_2}-63\vec{a_3}$.

(b) Odrediti koordinate vektora $\vec{a_2}$ u odnosu na bazu $\mathcal{B}'$.

Rj.-upute:

(a) Pokažimo da je skup $\mathcal{B}'$ linearno nezavisan skup tj. da je jedino rješenje sistema $\alpha\vec{b_1}+\beta\vec{b_2}+\gamma\vec{b_2}=\vec{0}$ (po nepoznatim $\alpha$, $\beta$ i $\gamma$) trivijalno rješenje

$$\alpha(15\vec{a_1}-\vec{a_2}+33\vec{a_3})+\beta(3\vec{a_1}+6\vec{a_3})+\gamma(-29\vec{a_1}+2\vec{a_2}-63\vec{a_3})=\vec{0}$$

$$\begin{aligned} 15\alpha+3\beta-29\gamma&=0\\ -\alpha\qquad\quad+2\gamma&=0\\ 33\alpha+6\beta-63\gamma&=0 \end{aligned} \qquad D=\begin{vmatrix} 15 & 3 & -29\\ -1 & 0 & 2\\ 33 & 6 & -63 \end{vmatrix}=\ldots=3\neq 0$$

$\Rightarrow$ skup $\mathcal{B}'$ je linearno nezavisan, a kako on ima tri elementa to generiše prostor $\mathbb{R}^3$ $\Rightarrow$ $\mathcal{B}'$ je baza prostora $\mathbb{R}^3$

(b) Odredimo skalare $\alpha$, $\beta$ i $\gamma$ t.d. $\alpha\vec{b_1}+\beta\vec{b_2}+\gamma\vec{b_3}=\vec{a_2}$

Iz (a) vidimo da se ovo svodi na sistem

$$\begin{aligned} 15\alpha+3\beta-29\gamma&=0\\ -\alpha\qquad\quad+2\gamma&=1\\ 33\alpha+6\beta-63\gamma&=0 \end{aligned}$$

ovaj sistem možemo riješiti npr. Kronecker-Kapelijevom metodom

$$\left[\begin{array}{ccc|c} 15 & 3 & -29 & 0\\ -1 & 0 & 2 & 1\\ 33 & 6 & -63 & 0 \end{array}\right]\sim\ldots\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5\\ 0 & 1 & 0 & 4\\ 0 & 0 & 1 & 3 \end{array}\right]$$

Koordinate vektora $\vec{a_2}$ u odnosu na bazu $\mathcal{B}'$ su $\begin{pmatrix}5\\4\\3\end{pmatrix}$.

(#) Vektor $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} \right\}$ ima koordinate $\begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix}$. Odrediti koordinate vektora $v$ u odnosu na bazu $\mathcal{B}' = \left\{ \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$.

Rj.-upute.

Posmatrajmo baze $\mathcal{B}$ i $\mathcal{B}'$. Nije teško vidjeti da je

$$\left.\begin{aligned} \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} &= 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} &= 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} &= (-1) \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \end{aligned} \right\} \ldots (*)$$

Kako su koordinate vektora $v$ u odnosu na bazu $\mathcal{B}$ $\begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix}$ to znači da je $v = 4 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} + 7 \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix}$.

Prema $(*)$ imamo

$$\begin{aligned} 4 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} &= 4 \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \qquad + 4 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ (-1) \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} &= (-1) \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + (-1) \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \\ 7 \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} &= (-7) \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \end{aligned}$$

Prema tome $v = (-4) \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Koordinate vektora $v$ u odnosu na bazu $\mathcal{B}'$ su $\begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix}$.

(#) Vektor $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B}=\left\{\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}0\\-1\\1\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right\}$ ima koordinate $\begin{pmatrix}5\\-1\\3\end{pmatrix}$. Odrediti koordinate vektora $v$ u odnosu na bazu $\mathcal{B}'=\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

Rj.-upute:

Posmatrajmo baze $\mathcal{B}$ i $\mathcal{B}'$. Nije teško vidjeti da je

$$\left.\begin{aligned}
\begin{pmatrix}2\\1\\2\end{pmatrix} &= 1\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}+1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}+1\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\\
\begin{pmatrix}0\\-1\\1\end{pmatrix} &= (-1)\begin{pmatrix}1\\1\\1\end{pmatrix}+1\begin{pmatrix}1\\0\\0\end{pmatrix}+2\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\\
\begin{pmatrix}0\\0\\2\end{pmatrix} &= 0\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}+0\begin{pmatrix}1\\0\\0\end{pmatrix}+2\begin{pmatrix}0\\0\\1\end{pmatrix}
\end{aligned}\right\}\ldots(*)$$

Kako su koordinate vektora $v\in\mathbb{R}^3$ u odnosu na bazu $\mathcal{B}$ $\begin{pmatrix}5\\-1\\3\end{pmatrix}$ to znači da je $v=5\begin{pmatrix}2\\1\\2\end{pmatrix}-1\begin{pmatrix}0\\-1\\1\end{pmatrix}+3\begin{pmatrix}0\\0\\2\end{pmatrix}$.

Prema $(*)$ imamo

$$\begin{aligned}
5\begin{pmatrix}2\\1\\2\end{pmatrix} &= 5\begin{pmatrix}1\\1\\1\end{pmatrix}+5\begin{pmatrix}1\\0\\0\end{pmatrix}+5\begin{pmatrix}0\\0\\1\end{pmatrix}\\
(-1)\begin{pmatrix}0\\-1\\1\end{pmatrix} &= 1\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}+(-1)\begin{pmatrix}1\\0\\0\end{pmatrix}+(-2)\begin{pmatrix}0\\0\\1\end{pmatrix}\\
3\begin{pmatrix}0\\0\\2\end{pmatrix} &= \qquad\qquad\qquad\qquad\qquad 6\begin{pmatrix}0\\0\\1\end{pmatrix}
\end{aligned}$$

Prema tome $v=6\begin{pmatrix}1\\1\\1\end{pmatrix}+4\begin{pmatrix}1\\0\\0\end{pmatrix}+9\begin{pmatrix}0\\0\\1\end{pmatrix}$

Koordinate vektora $v$ u odnosu na bazu $\mathcal{B}'$ su $\begin{pmatrix}6\\4\\9\end{pmatrix}$.

(#) Vektor $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ ima koordinate $\begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$. Odrediti koordinate vektora $v$ u odnosu na bazu $\mathcal{B}' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Rj.-upute:

Posmatrajmo baze $\mathcal{B}$ i $\mathcal{B}'$. Nije teško vidjeti da je

$$\left. \begin{aligned} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} &= 0 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} &= (-1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} &= 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned} \right\} \cdots (*)$$

Kako su koordinate vektora $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B}$ $\begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$ to znači da je $v = 7 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Sad prema $(*)$ imamo

$$\begin{aligned} 7 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} &= 7 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ -3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} &= 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-3) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ 5 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} &= 5 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-5) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

Prema tome $v = 8 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Koordinate vektora $v$ u odnosu na bazu $\mathcal{B}'$ su $\begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix}$.

㊉ Bez upotrebe H'Lopitalovog pravila izračunati limese a) $\lim\limits_{x\to 3}\frac{3x^2-10x+3}{2x^2-7x+3}$;

b) $\lim\limits_{x\to\frac{\pi}{2}}\frac{\cos^2 x}{1+\sin^3 x}$.

Rj.-upute:

a) $ax^2+bx+c=a(x-x_1)(x-x_2)$

$$\lim_{x\to 3}\frac{3x^2-10x+3}{2x^2-7x+3}=\lim_{x\to 3}\frac{3\cancel{(x-3)}(x-\frac{1}{3})}{2(x-\frac{1}{2})\cancel{(x-3)}}=\lim_{x\to 3}\frac{3x-1}{2x-1}=\frac{8}{5}$$

b)

$$\cos^2 x=1-\sin^2 x=(1-\sin x)(1+\sin x)$$

$$1+\sin^3 x=(1+\sin x)(1-\sin x+\sin^2 x)$$

$$\lim_{x\to-\frac{\pi}{2}}\frac{\cos^2 x}{1+\sin^3 x}=\lim_{x\to-\frac{\pi}{2}}\frac{(1-\sin x)\cancel{(1+\sin x)}}{\cancel{(1+\sin x)}(1-\sin x+\sin^2 x)}=\frac{1-(-1)}{1-(-1)+1}=\frac{2}{3}$$

(#) Bez upotrebe H'Lopitalovog pravila izračunati limese a) $\lim\limits_{x\to 7}\dfrac{2x^2-13x-7}{-2x^2+11x+21}$

b) $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{1-\sin^3 x}{\cos^2 x}$ .

Rj.-upute:

a)

$$ax^2+bx+c=a(x-x_1)(x-x_2)$$

$$2x^2-13x-7=2(x-7)\left(x+\frac{1}{2}\right)$$

$$-2x^2+11x+21=(-2)\left(x+\frac{3}{2}\right)(x-7)$$

$$\lim_{x\to 7}\frac{2x^2-13x-7}{-2x^2+11x+21}=\lim_{x\to 7}\frac{2\cancel{(x-7)}\left(x+\frac{1}{2}\right)}{(-2)\left(x+\frac{3}{2}\right)\cancel{(x-7)}}=\lim_{x\to 7}\frac{2x+1}{-2x-3}=\frac{15}{-17}$$

b)

$$1-\sin^3 x=(1-\sin x)(1+\sin x+\sin^2 x)$$

$$\cos^2 x=1-\sin^2 x=(1-\sin x)(1+\sin x)$$

$$\lim_{x\to\frac{\pi}{2}}\frac{1-\sin^3 x}{\cos^2 x}=\lim_{x\to\frac{\pi}{2}}\frac{\cancel{(1-\sin x)}(1+\sin x+\sin^2 x)}{\cancel{(1-\sin x)}(1+\sin x)}=\frac{1+1+1}{1+1}=\frac{3}{2}$$

(#) Bez upotrebe H'Lopitalovog pravila izračunati limese

a) $\lim\limits_{x\to 1} \dfrac{5x^2-3x-2}{7x^2-10x+3}$

b) $\lim\limits_{x\to 0} \dfrac{1-\cos^3 x}{\sin^2 x}$.

Rj.-upute:

a) $ax^2+bx+c=a(x-x_1)(x-x_2)$

$$\lim_{x\to 1}\frac{5x^2-3x-2}{7x^2-10x+3}=\lim_{x\to 1}\frac{5\cancel{(x-1)}(x+\frac{2}{5})}{7(x-\frac{3}{7})\cancel{(x-1)}}=\lim_{x\to 1}\frac{5x+2}{7x-3}=\frac{7}{4}$$

b)

$$1-\cos^3 x=(1-\cos x)(1+\cos x+\cos^2 x)$$

$$\sin^2 x=1-\cos^2 x=(1-\cos x)(1+\cos x)$$

$$\lim_{x\to 0}\frac{1-\cos^3 x}{\sin^2 x}=\lim_{x\to 0}\frac{\cancel{(1-\cos x)}(1+\cos x+\cos^2 x)}{\cancel{(1-\cos x)}(1+\cos x)}=\frac{1+1+1}{1+1}=\frac{3}{2}$$

(#) Bez upotrebe H'Lopitalovog pravila izračunati limese a) $\lim\limits_{x\to 5}\frac{2x^2-11x+5}{3x^2-14x-5}$ ;

b) $\lim\limits_{x\to \pi}\frac{\sin^2 x}{1+\cos^3 x}$ ;

Rj.–upute:

a)

$$ax^2+bx+c=a(x-x_1)(x-x_2)$$

$$2x^2-11x+5=2(x-5)\left(x-\frac{1}{2}\right)$$

$$3x^2-14x-5=3(x-5)\left(x+\frac{1}{3}\right)$$

$$\lim_{x\to 5}\frac{2x^2-11x+5}{3x^2-14x-5}=\lim_{x\to 5}\frac{2(x-5)\left(x-\frac{1}{2}\right)}{3(x-5)\left(x+\frac{1}{3}\right)}=\lim_{x\to 5}\frac{2x-1}{3x+1}=\frac{9}{16}$$

b)

$$\sin^2 x=1-\cos^2 x=(1-\cos x)(1+\cos x)$$

$$1+\cos^3 x=(1+\cos x)(1-\cos x+\cos^2 x)$$

$$\lim_{x\to\pi}\frac{\sin^2 x}{1+\cos^3 x}=\lim_{x\to\pi}\frac{(1-\cos x)\cancel{(1+\cos x)}}{\cancel{(1+\cos x)}(1-\cos x+\cos^2 x)}=\frac{1-(-1)}{1-(-1)+1}=\frac{2}{3}$$

(#) Bez upotrebe H'Lopitalovog pravila izračunati sljedeće limese

a) $\lim\limits_{x\to -2} \frac{-5x^2-30x-40}{-3x^2+6x+24}$

c) $\lim\limits_{x\to -6} \frac{3x^2+12x-36}{2x^2+10x-12}$

b) $\lim\limits_{x\to -4} \frac{-4x^2-12x+16}{-2x^2-10x-8}$

d) $\lim\limits_{x\to -8} \frac{5x^2+35x-40}{-2x^2-6x+80}$

Rj.

a) $$\lim_{x\to -2} \frac{-5x^2-30x-40}{-3x^2+6x+24} = \lim_{x\to -2} \frac{(-5)\cancel{(x+2)}(x+4)}{(-3)\cancel{(x+2)}(x-4)} = \frac{-5\cdot 2}{-3\cdot(-6)} = -\frac{10}{18} = -\frac{5}{9}$$

b) $$\lim_{x\to -4} \frac{-4x^2-12x+16}{-2x^2-10x-8} = \lim_{x\to -4} \frac{(-4)\cancel{(x+4)}(x-1)}{(-2)\cancel{(x+4)}(x+1)} = \frac{(-4)\cdot(-5)}{(-2)(-3)} = \frac{20}{6} = \frac{10}{3}$$

c) $$\lim_{x\to -6} \frac{3x^2+12x-36}{2x^2+10x-12} = \lim_{x\to -6} \frac{3\cancel{(x+6)}(x-2)}{2\cancel{(x+6)}(x-1)} = \frac{3\cdot \overset{-4}{\cancel{(-8)}}}{\cancel{2}\cdot(-7)} = \frac{12}{7}$$

d) $$\lim_{x\to -8} \frac{5x^2+35x-40}{-2x^2-6x+80} = \lim_{x\to -8} \frac{5\cancel{(x+8)}(x-1)}{(-2)\cancel{(x+8)}(x-5)} = \frac{5\cdot(-9)}{(-2)(-13)} = -\frac{45}{26}$$

(#) Bez upotrebe H'Lopitalovog pravila izračunati limese

$$\text{(a)}\quad \lim_{x\to 4}\frac{2-\sqrt{x}}{3-\sqrt{2x+1}}\,; \qquad \text{(b)}\quad \lim_{x\to 9}\frac{4-\sqrt{2x-2}}{3-\sqrt{x}}$$

Rj.

$$\text{(a)}\quad \lim_{x\to 4}\frac{2-\sqrt{x}}{3-\sqrt{2x+1}}\underbrace{\left(=\frac{0}{0}\right)}_{\text{neodređen izraz}} = \lim_{x\to 4}\frac{(2-\sqrt{x})(3+\sqrt{2x+1})}{(3-\sqrt{2x+1})(3+\sqrt{2x+1})} =$$

$$= \lim_{x\to 4}\frac{(2-\sqrt{x})(3+\sqrt{2x+1})}{9-2x-1}\underbrace{\left(=\frac{0}{0}\right)}_{\text{neodređen izraz}} = \lim_{x\to 4}\frac{(2-\sqrt{x})(3+\sqrt{2x+1})(2+\sqrt{x})}{(8-2x)(2+\sqrt{x})} =$$

$$= \lim_{x\to 4}\frac{(4-x)(3+\sqrt{2x+1})}{2(4-x)(2+\sqrt{x})} = \frac{1}{2}\lim_{x\to 4}\frac{3+\sqrt{2x+1}}{2+\sqrt{x}} = \frac{1}{2}\cdot\frac{6}{4} = \frac{3}{4}$$ traženo rješenje

$$\text{(b)}\quad \lim_{x\to 9}\frac{4-\sqrt{2x-2}}{3-\sqrt{x}}\underbrace{\left(=\frac{0}{0}\right)}_{\text{neodređen izraz}} = \lim_{x\to 9}\frac{(4-\sqrt{2x-2})(3+\sqrt{x})}{(3-\sqrt{x})(3+\sqrt{x})} =$$

$$= \lim_{x\to 9}\frac{(4-\sqrt{2x-2})(3+\sqrt{x})}{9-x}\underbrace{\left(=\frac{0}{0}\right)}_{\text{neodređen izraz}} = \lim_{x\to 9}\frac{(4-\sqrt{2x-2})(3+\sqrt{x})(4+\sqrt{2x-2})}{(9-x)(4+\sqrt{2x-2})}$$

$$= \lim_{x\to 9}\frac{(3+\sqrt{x})(16-2x+2)}{(9-x)(4+\sqrt{2x-2})} = \lim_{x\to 9}\frac{(3+\sqrt{x})\,2\cancel{(9-x)}}{\cancel{(9-x)}(4+\sqrt{2x-2})} = 2\cdot\frac{6}{4+4} = \frac{2\cdot 6}{8} = \frac{6}{4}$$

$$= \frac{3}{2}$$ traženo rješenje

(#) Odrediti prvi izvod f-je

(a) $y = \ln \frac{x^2-1}{x+1} + \operatorname{arc\,tg} x^2$

(b) $y = \ln \frac{x}{x-1} + \arcsin x^2$

c) $y = \ln \frac{x^2}{x+1} + \operatorname{tg} x^2$

Rj. - upute

(a) $\left(\ln \frac{x^2-1}{x+1}\right)' = \ldots = \frac{1}{x-1}$ ; $\left(\operatorname{arc\,tg} x^2\right)' = \frac{2x}{x^4+1}$

$$y' = \frac{1}{x-1} + \frac{2x}{x^4+1}$$

b) $\left(\ln \frac{x}{x-1}\right)' = \ldots = \frac{-1}{x^2-x}$ ; $\left(\arcsin x^2\right)' = \frac{2x}{\sqrt{1-x^4}}$

$$y' = \frac{-1}{x^2-x} + \frac{2x}{\sqrt{1-x^4}}$$

(c) $\left(\ln \frac{x^2}{x+1}\right)' = \ldots = \frac{x+2}{x^2+1}$ ; $\operatorname{tg} x^2 = \frac{2x}{\cos^2 x}$

$$y' = \frac{x+2}{x^2+1} + \frac{2x}{\cos^2 x}$$

(#) Odrediti jednačinu tangente i normale

(a) na krivu $x^2+y^2-2x+4y-3=0$

(b) na krivu $x^2+y^2+4x-2y+3=0$

u tačkama u kojima kriva siječe x-osu.

Rj. -upute:

(a) Za $y=0$ imamo $x^2-2x-3=0$

$(x-3)(x+1)=0$

Kriva siječe x-osu u tačkama $A(3;0)$ i $B(-1;0)$.

$$x^2+y^2-2x+4y-3=0 \quad /'$$

$$2x+2yy'-2+4y'=0$$

$$2yy'+4y'=2-2x$$

$$(2y+4)y'=2-2x \quad /:2$$

$$y'=\frac{1-x}{2+y}$$

$$y'(A)=-1, \quad y'(B)=1$$

$y-y_1=k(x-x_1)$ $A(3;0)$

$y=(-1)(x-3)$

$y=-x+3$

$y-y_1=-\frac{1}{k}(x-x_1)$ $A(3;0)$

$y=1(x-3)$

$y=x-3$

Jednačina tangente na datu krivu u tački $A(3;0)$ je $y=-x+3$ a jednačina normale je $y=x-3$.

$y-y_1=k(x-x_1)$ $B(-1;0)$

$y=1(x+1)$

$y-y_1=-\frac{1}{k}(x-x_1)$ $B(-1;0)$

$y=-1(x+1)$

Jednačina tangente na datu krivu u tački B je $y=x+1$ a jednačina normale je $y=-x-1$.

(b) $x^2+y^2+4x-2y+3=0$

za $y=0$ $\quad x^2+4x+3=0$

$$(x+3)(x+1)=0$$

Kriva siječe x-osu u tačkama $C(-3;0)$ i $D(-1;0)$.

$$x^2+y^2+4x-2y+3=0 \quad /'$$

$$2x+2yy'+4-2y'=0$$

$$2yy'-2y'=-2x-4 \quad /:2$$

$$(y-1)y'=-x-2$$

$$y'=\frac{-x-2}{y-1}$$

$$y'(C)=\frac{3-2}{0-1}=-1$$

$$y'(D)=\frac{1-2}{-1}=1$$

$$y-y_1=k(x-x_1)$$

$C$: $\quad y=(-1)(x+3)=-x-3$

$D$: $\quad y=1(x+1)=x+1$

normale

za $C$ koeficijent normale je $1$

za tačku $D$ koeficijent normale je $-\frac{1}{k}=-1$

Jednačine tangente na datu krivu u tačkama $C$ i $D$ su redom

$y=-x-3$ ; $\quad y=x+1$.

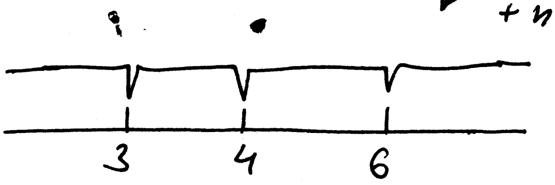
Jednačine normale na datu krivu u tačkama $C$ i $D$ su redom

$y=x+3$ ; $\quad y=-x-1$

(#) Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $y = \dfrac{(x-3)^3}{(x-4)^2}$ .

Rj. - upute:

$$y' = \frac{(x-3)^2(x-6)}{(x-4)^3}$$



↙ prekidi y + nule y'

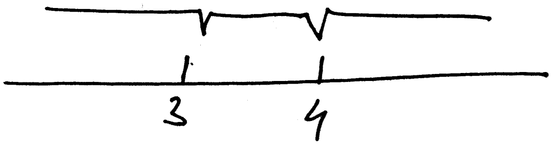
| x | $(-\infty, 3)$ | $(3, 4)$ | $(4, 6)$ | $(6, +\infty)$ |
|---|---|---|---|---|
| y' | + | + | − | + |
| y | ↗ | ↗ | ↘ | ↗ |

MIN

tabela rasta i opadanja

$$\text{MIN}\left(6, \frac{27}{4}\right)$$

$$y'' = 6\,\frac{x-3}{(x-4)^4}$$



↙ prekidi y + nule y''

| x | $(-\infty, 3)$ | $(3, 4)$ | $(4, +\infty)$ |
|---|---|---|---|
| y'' | − | + | + |
| y | ∩ | ∪ | ∪ |

P.T.

tabela konveksnosti i konkavnosti

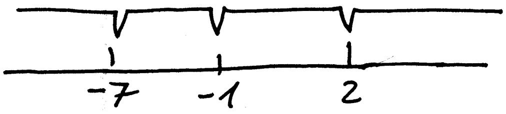
P.T. (3, 0)

(#) Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $y = \frac{(x-2)^3}{(x+1)^2}$.

Rj.-upute:

↙ prekidi y + nule y'

$$y' = \frac{(x-2)^2(x+7)}{(x+1)^3}$$



| x | $(-\infty,-7)$ | $(-7,-1)$ | $(-1,2)$ | $(2,+\infty)$ |
|---|---|---|---|---|
| y' | + | − | + | + |
| y | ↗ | ↘ | ↗ | ↗ |

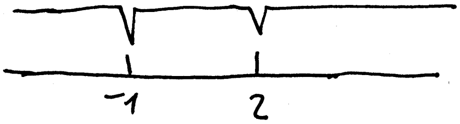
MAX

tabela rasta i opadanja

MAX$(-7, -\frac{81}{4})$

↙ prekidi y +nule y''

$$y'' = 54\,\frac{x-2}{(x+1)^4}$$



| x | $(-\infty,-1)$ | $(-1,2)$ | $(2,+\infty)$ |
|---|---|---|---|
| y'' | − | − | + |
| y | ∩ | ∩ | ∪ |

P.T.

tabela konveksnosti i konkavnosti

P.T.$(2, 0)$

## (#) Odrediti kosu asimptotu sljedećih f-ja

a) $Y = \frac{x^4+1}{x^3-1}$ b) $Y = \frac{2x^2-3x+4}{x-2}$

c) $Y = \frac{3x^4-x}{x^3+2}$ d) $Y = \frac{2x^3+4}{x^2-x+1}$

Rj. -upute:
Kosu asimptotu f-je $f(x)$ tražimo u obliku $y=kx+n$ gdje je

$$k=\lim_{x\to\infty}\frac{f(x)}{x}, \quad n=\lim_{x\to\infty}\left[f(x)-kx\right]$$

a) $$k=\lim_{x\to\infty}\frac{x^4+1 \ /:x^4}{x^4-x \ /:x^4}=\lim_{x\to\infty}\frac{1+\frac{1}{x^4}}{1-\frac{1}{x^3}}=1$$

$$n=\lim_{x\to\infty}\left(\frac{x^4+1}{x^3-1}-x\right)=\lim_{x\to\infty}\frac{x^4+1-x+}{x^3-1}=\lim_{x\to\infty}\frac{x+1}{x^3-1}=0$$

$\Uparrow$ $Y=x$ je tražena kosa asimptota

b) $$k=\lim_{x\to\infty}\frac{2x^2-3x+4 \ /:x^2}{x^2-2x \ /:x^2}=2$$

$$n=\lim_{x\to\infty}\left(\frac{2x^2-3x+4}{x-2}-2x\right)=\lim_{x\to\infty}\frac{\cancel{2x^2}-3x+4\cancel{-2x^2}+4x}{x-2}=1$$

$\Uparrow$ $Y=2x+1$ je tražena kosa asimptota

c) $$k=\lim_{x\to\infty}\frac{3x^4-x}{x^4+2x}=3$$

$$n=\lim_{x\to\infty}\left(\frac{3x^4-x}{x^3+2}-3x\right)=\lim_{x\to\infty}\frac{3x^4-x-3x^4-6x}{x^3+2}=0$$

$\Uparrow$ $Y=3x$ je tražena K. A.

d) $$k=\lim_{x\to\infty}\frac{2x^3+4}{x^3-x^2+x}=2$$

$$n=\lim_{x\to\infty}\left(\frac{2x^3+4}{x^2-x+1}-2x\right)=\lim_{x\to\infty}\frac{\cancel{2x^3}+4\cancel{-2x^3}+2x^2-2x}{x^2-x+1}=2$$

$\Uparrow$ $Y=2x+2$ je tražena kosa asimptota

(#) Ispitati i nacrtati graf f-je $y=\frac{x-2}{x^2-8x+16}$

Rj.-upute:

DEFINICIONO PODRUČJE

$y=\frac{x-2}{(x-4)^2}$ $\quad x\neq 4$

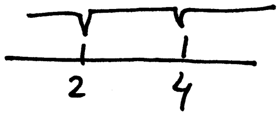
$D: x\in\mathbb{R}\setminus\{4\}$

$x\in(-\infty,4)\cup(4,+\infty)$

ZNAK, NULE, PRESJEK SA y-OSOM

$(2;0)$ je nula f-je

$(0;-\frac{1}{8})$ je presjek sa y-osom



← prekid y + nule y

| x | $(-\infty,2)$ | $(2,4)$ | $(4,+\infty)$ |
|---|---|---|---|
| y | − | + | + |

znak f-je

PARNOST (NEPARNOST), PERIODIČNOST

Definiciono područje nije simetrično pa f-ja nije ni parna ni neparna

PONAŠANJE NA KRAJEVIMA INTERVALA DEFINISANOSTI I ASIMPTOTE

vertikalna asimptota F-ja ima prekid za x=4

$$\lim_{x\to 4-0} f(x)=\lim_{x\to 4-0}\frac{x-2}{(x-4)^2}=\frac{4-0-2}{+0}=+\infty$$

$$\lim_{x\to 4+0} f(x)=\lim_{x\to 4+0}\frac{x-2}{(x-4)^2}=\frac{4+0-2}{+0}=+\infty$$

$\Rightarrow x=4$ je V.A.

horizontalna asimptota

$$\lim_{x\to\mp\infty} f(x)=\lim_{x\to\mp\infty}\frac{x-2 \quad /:x}{x^2-8x+16 \quad /:x}=0$$

$\Rightarrow y=0$ je H.A.

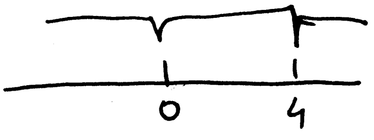
f-ja nema kosu asimptotu

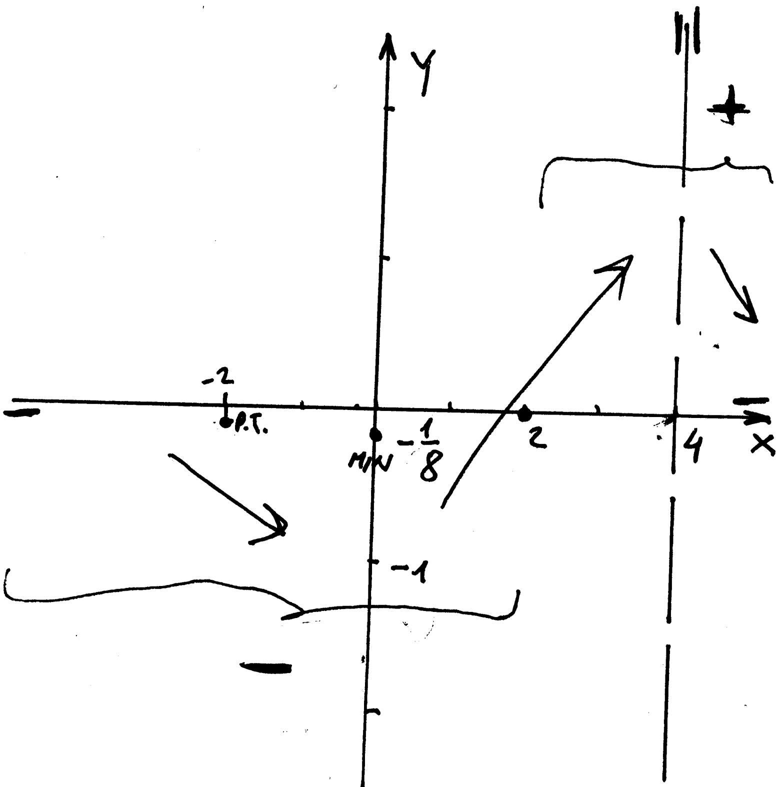
Poslije ovog koraka počinjemo skicirati graf f-je.

RAST I OPADANJE

$y'=-\frac{x}{(x-4)^3}$

prekidi y + nule y' →





| x | $(-\infty, 0)$ | $(0, 4)$ | $(4, +\infty)$ |
|---|---|---|---|
| $y'$ | $-$ | $+$ | $-$ |
| $y$ | ↘ | ↗ | ↘ |

MIN

intervali rasta i opadanja

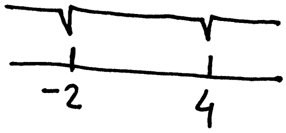
$f(0) = -\frac{1}{8}$

EKSTREMI F-JE

Na osnovu tabele rasta i opadanja f-ja ima minimum u tački $(0; -\frac{1}{8})$.

PREVOJNE TAČKE I INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$$y'' = \frac{2(x+2)}{(x-4)^4}$$



✓ prekidi y + nule y''

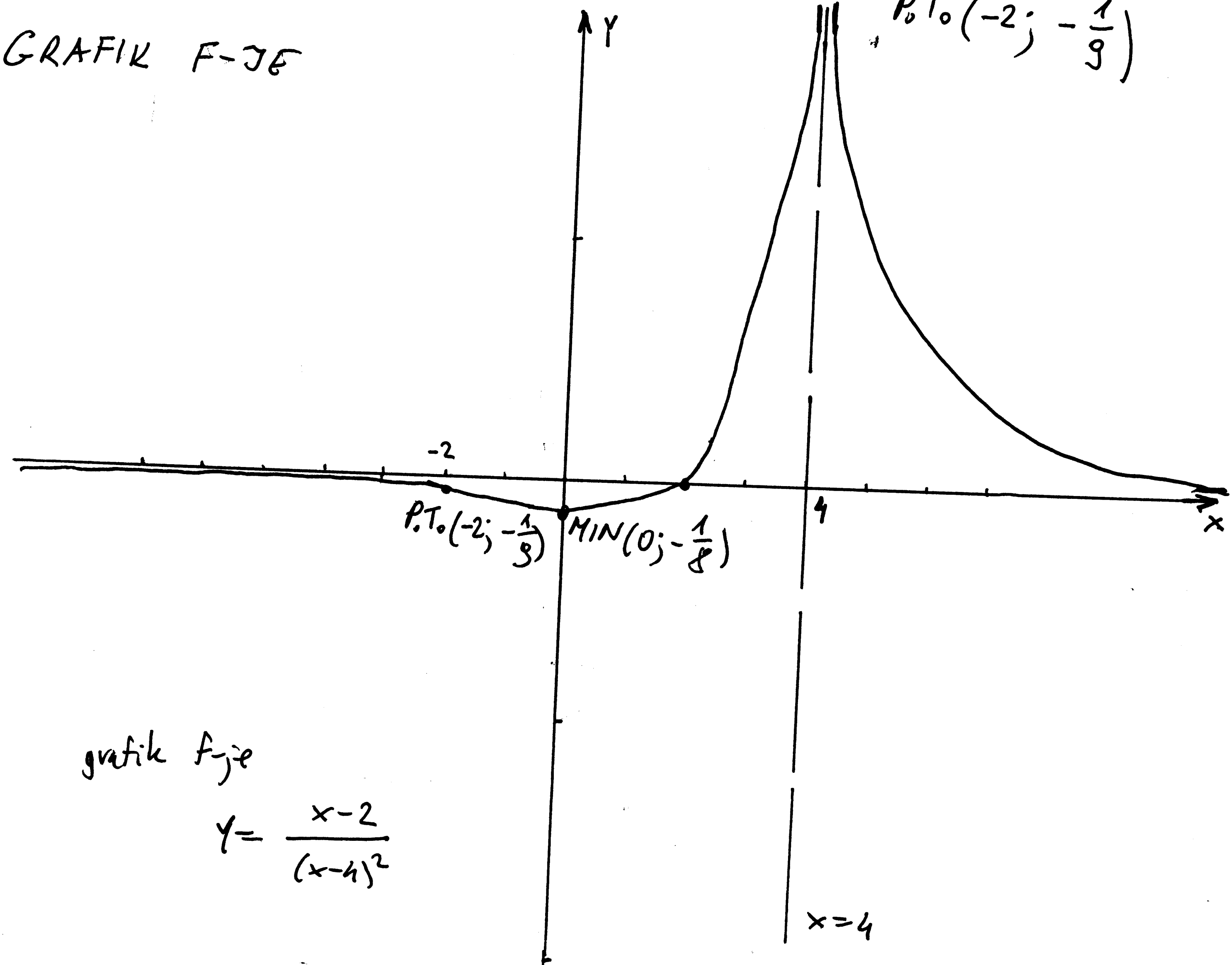
| x | $(-\infty, -2)$ | $(-2, 4)$ | $(4, +\infty)$ |
|---|---|---|---|
| $y''$ | $-$ | $+$ | $+$ |
| $y$ | $\cap$ | $\cup$ | $\cup$ |

$P_0T_0$

tabela konveksn. i konkavnosti

$P_0T_0(-2; -\frac{1}{9})$

GRAFIK F-JE



grafik f-je

$$y = \frac{x-2}{(x-4)^2}$$

## (#) Ispitati i nacrtati graf f-je $Y = \frac{x-5}{x^2-2x+1}$

Rj. -upute:

DEFINICIONO PODRUČJE

$Y = \frac{x-5}{(x-1)^2}$ $\quad (x-1)^2 \neq 0$

$x-1 \neq 0$

$x \neq 1$

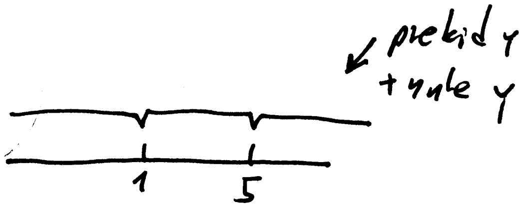
$\mathcal{D}: x \in \mathbb{R} \setminus \{1\}$

$x \in (-\infty, 1) \cup (1, +\infty)$

PARNOST, NEPARNOST, PERIODIČNOST

$\mathcal{D}$ nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična



ZNAK, NULE, PRESJEK SA Y-osom

(5; 0) je nula f-je

(0; -5) je presjek sa Y-osom

| x | $(-\infty, 1)$ | $(1, 5)$ | $(5, +\infty)$ |
|---|---|---|---|
| Y | − | − | + |

Znak f-je

PONAŠANJE NA KRAJEVIMA INTERVALA DEFINISANOSTI I ASIMPTOT

vertikalna asimptota F-ja ima prekid za $x=1$

$$\lim_{x\to1-0} f(x) = \lim_{x\to1-0} \frac{x-5}{(x-1)^2} = \frac{1-0-5}{+0} = -\infty$$

$$\lim_{x\to1+0} f(x) = \lim_{x\to1+0} \frac{x-5}{(x-1)^2} = \frac{1+0-5}{+0} = -\infty$$

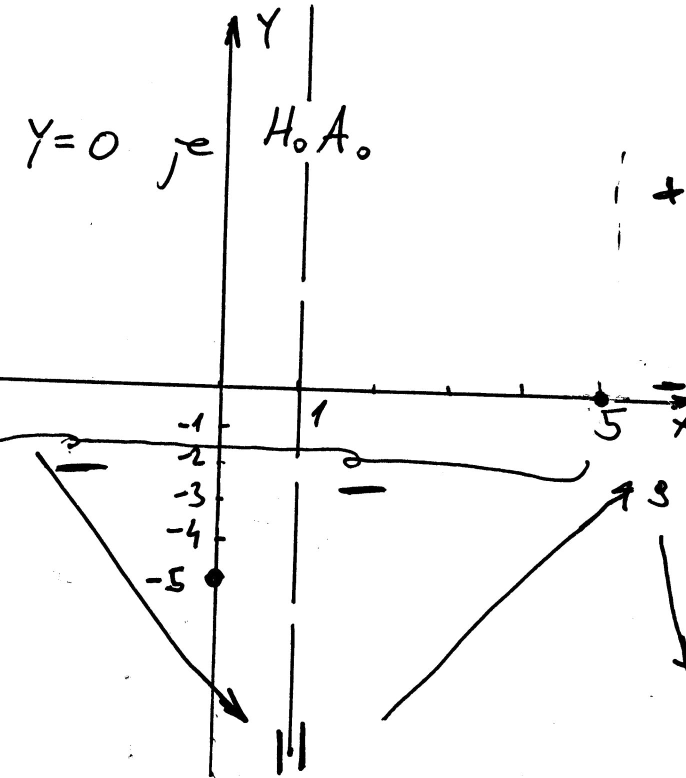
$\Rightarrow x=1$ je V. A.

horizontalna asimptota

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x-5 \quad /:x}{x^2-2x+1 \quad /:x} = 0 \Rightarrow Y=0 \text{ je H. A.}$$

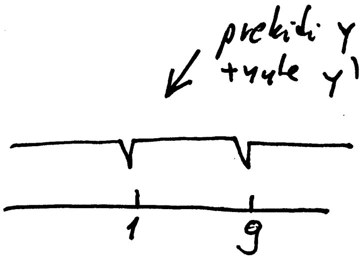
f-ja nema kosu asimptotu

Poslije ovog koraka počinjemo skicirati graf f-je.



RAST I OPADANJE

$$Y' = -\frac{x-9}{(x-1)^3}$$



| x | $(-\infty, 1)$ | $(1, 9)$ | $(9, +\infty)$ |
|---|---|---|---|
| $y'$ | $-$ | $+$ | $-$ |
| $y$ | ↘ | ↗ | ↘ |

MAX

tabela rasta i opadanja

$$f(9) = \frac{9-5}{(9-1)^2} = \frac{4}{64}$$

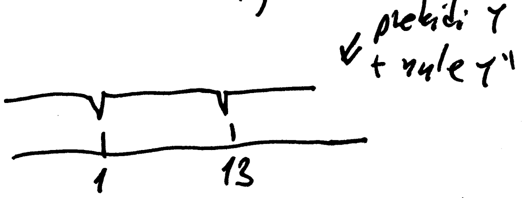
$$f(9) = \frac{1}{16}$$

EKSTREMI F-JE

Na osnovu tabele rasta i opadanja f-ja ima ekstrem i to maksimum u tački $M(9; \frac{1}{16})$

PREVOJNE TAČKE I INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$$y'' = 2\,\frac{x-13}{(x-1)^4}$$



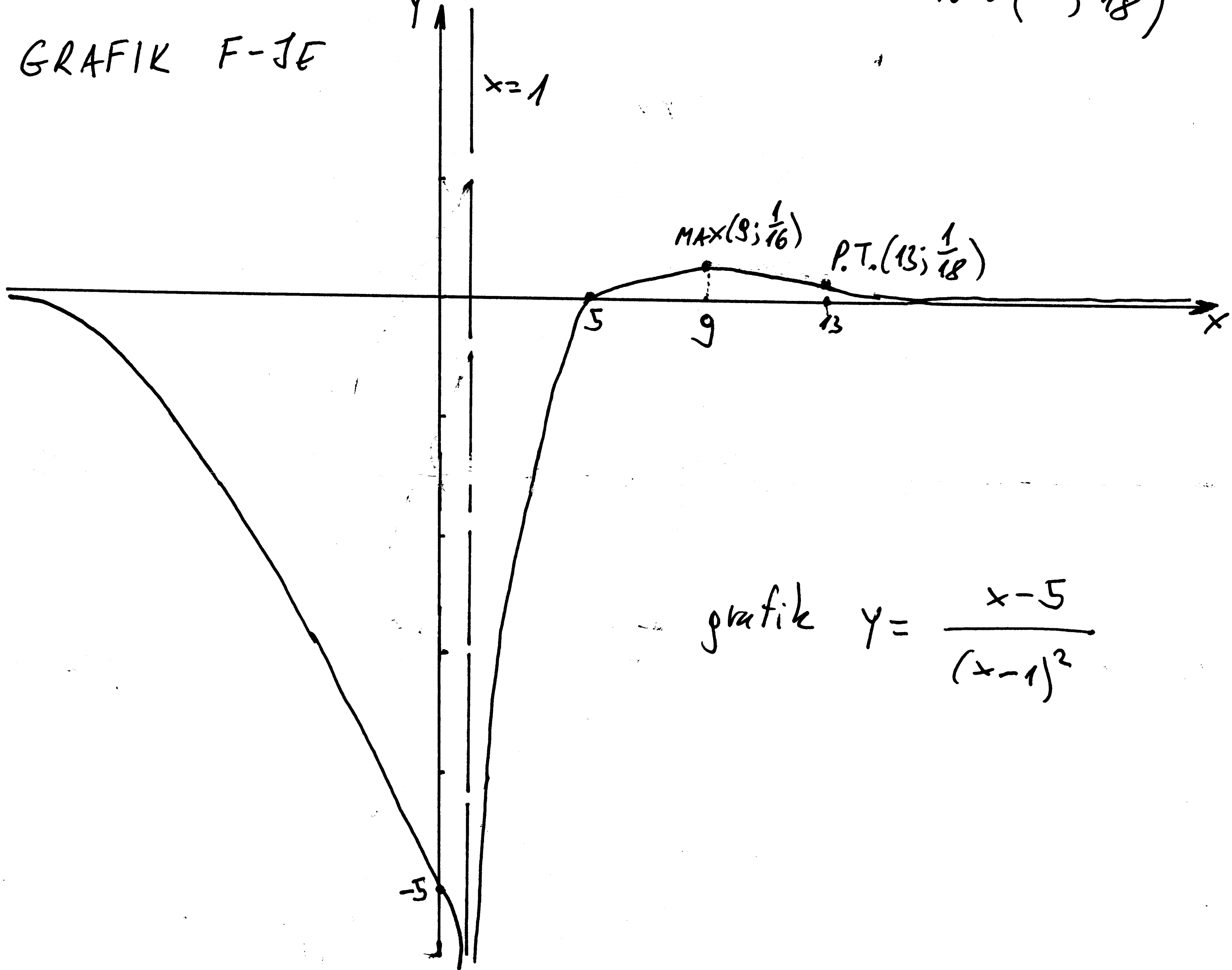
| x | $(-\infty, 1)$ | $(1, 13)$ | $(13, +\infty)$ |
|---|---|---|---|
| $y''$ | $-$ | $-$ | $+$ |
| $y$ | ∩ | ∩ | ∪ |

P.T₀

tabela konveksnosti i konkavnosti

$P_0T_0\left(13, \frac{1}{18}\right)$

GRAFIK F-JE



grafik $y = \frac{x-5}{(x-1)^2}$

(#) Ispitati i nacrtati graf f-je $y=\frac{x-3}{x^2-4x+4}$

Rj.-upute

DEFINICIONO PODRUČJE

$y=\frac{x-3}{(x-2)^2}$ $\quad x\neq 2$

$\mathcal{D}: x\in\mathbb{R}\setminus\{2\}$

$x\in(-\infty,2)\cup(2,+\infty)$

PARNOST (NEPARNOST), PERIODIČNOST

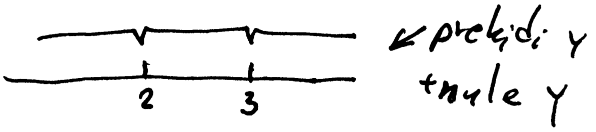
Definiciono područje nije simetrično pa f-ja nije ni parna ni neparna

F-ja nije periodična

ZNAK, NULE, PRESJEK SA Y-OSOM

$(3;0)$ je nula f-je

$(0;-\frac{3}{4})$ je presjek f-je sa Y-osom



| x | $(-\infty,2)$ | $(2,3)$ | $(3,+\infty)$ |
|---|---|---|---|
| y | − | − | + |

znak f-je

PONAŠANJE NA KRAJEVIMA INTERVALA DEFINISANOSTI I ASIMPTOTE

vertikalna asimptota F-ja ima prekid za $x=2$

$$\left.\begin{aligned}\lim_{x\to 2-0} f(x)&=\lim_{x\to 2-0}\frac{x-3}{(x-2)^2}=\frac{2-0-3}{+0}=-\infty\\ \lim_{x\to 2+0} f(x)&=\lim_{x\to 2+0}\frac{x-3}{(x-2)^2}=\frac{2+0-3}{+0}=-\infty\end{aligned}\right\}\Rightarrow x=2 \text{ je V.A.}$$

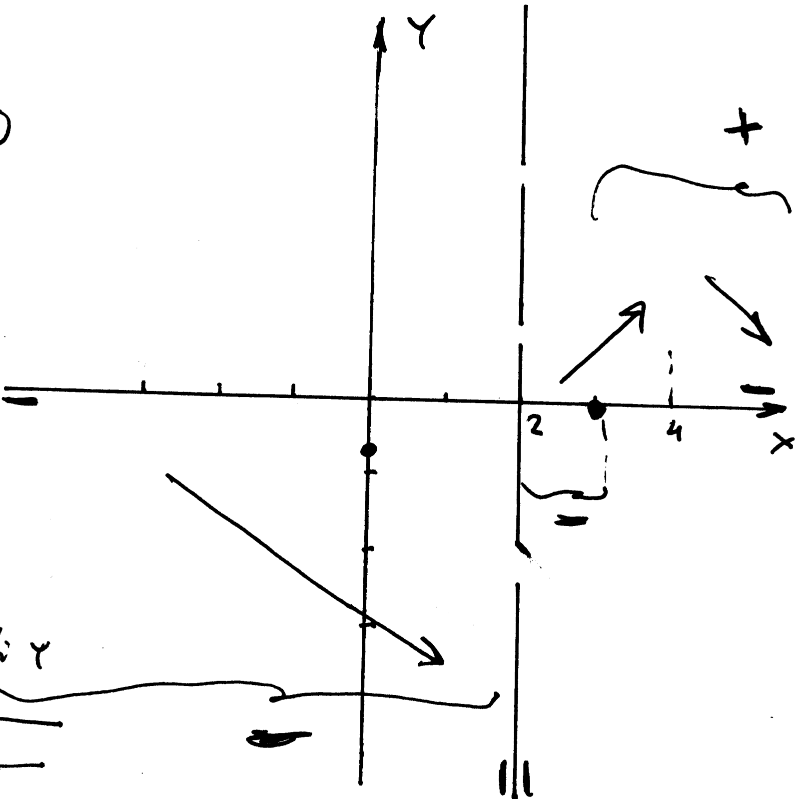
horizontalna asimptota

$$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{x-3}{x^2-4x+4}\begin{matrix}/:x\\/:x\end{matrix}=0$$

$\Rightarrow y=0$ je H.A.

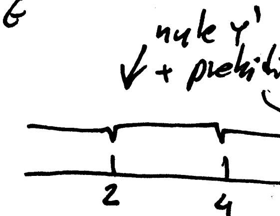
f-ja nema kosu asimptotu.

Poslije ovog koraka počinjemo sa skiciranjem grafa f-je.



RAST I OPADANJE

$y'=-\frac{x-4}{(x-2)^3}$



| x | $(-\infty, 2)$ | $(2, 4)$ | $(4, +\infty)$ |
|---|---|---|---|
| Y' | − | + | − |
| Y | ↘ | ↗ | ↘ |
| | | MAX | |

tabela rasta i opadanja

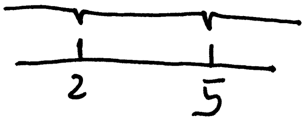
$$f(4) = \frac{1}{4}$$

EKSTREMI F-JE

Na osnovu tabele rasta i opadanja f-ja ima maksimum u tački $M(4; \frac{1}{4})$.

PREVOJNE TAČKE I INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$$Y'' = \frac{2(x-5)}{(x-2)^4}$$



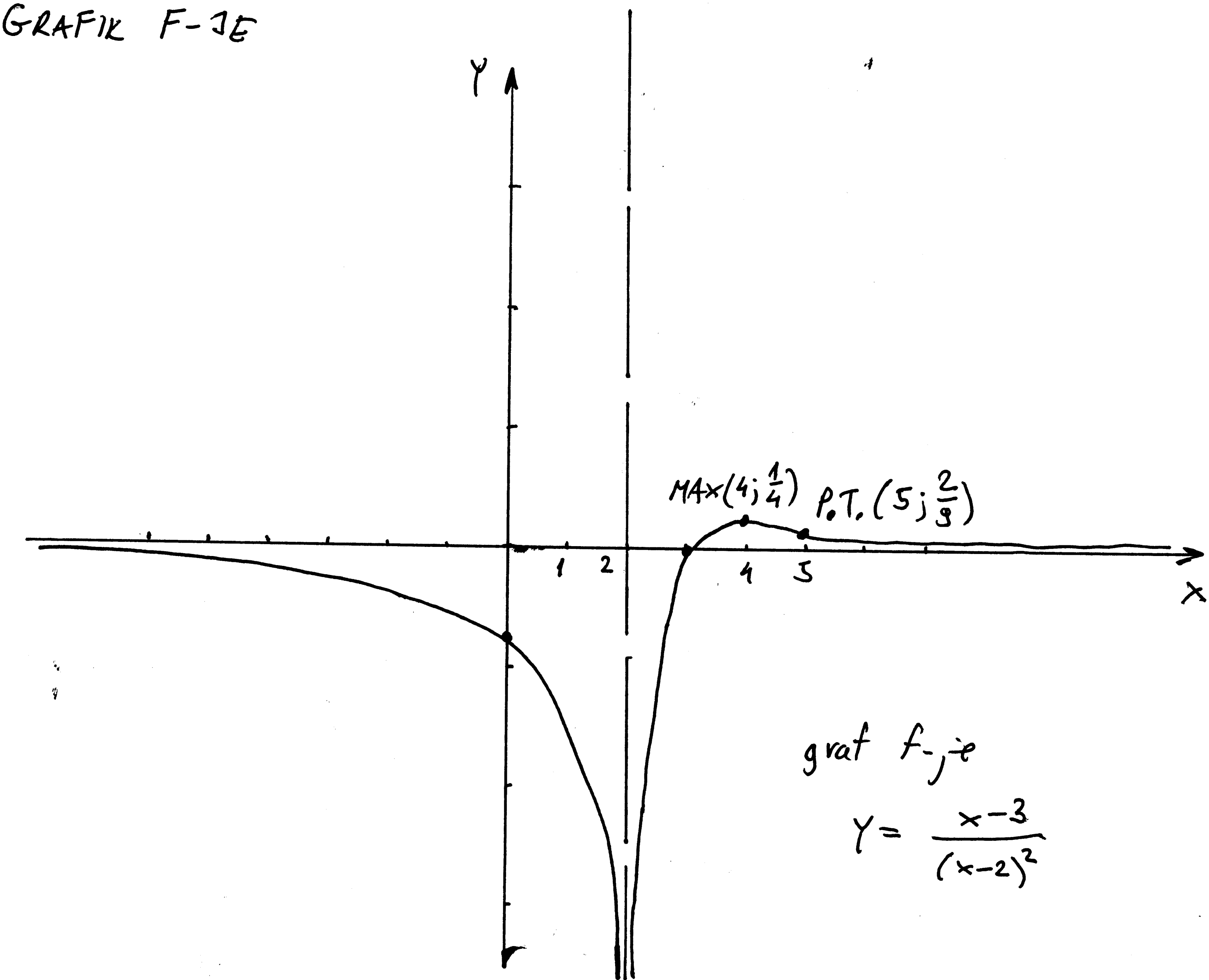
prekidi Y + nule Y''

| x | $(-\infty, 2)$ | $(2, 5)$ | $(5, +\infty)$ |
|---|---|---|---|
| Y'' | − | − | + |
| Y | ∩ | ∩ | ∪ |
| | | P.T. | |

tabela konv. i konk.

$$P.T.(5, \frac{2}{9})$$

GRAFIK F-JE



graf f-je

$$Y = \frac{x-3}{(x-2)^2}$$

(#) Ispitati i nacrtati grafik f-je $Y = \dfrac{x-1}{x^2-10x+25}$

Rj.-upute

DEFINICIONO PODRUČJE

$Y = \dfrac{x-1}{(x-5)^2}$ $\quad (x-5)^2 \neq 0$

$x-5 \neq 0$

$x \neq 5$

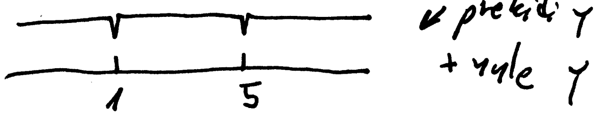
$\mathcal{D}: x \in R \setminus \{5\}$

$x \in (-\infty, 5) \cup (5, +\infty)$

ZNAK, NULE, PRESJEK SA Y-OSOM

$(1; 0)$ je nula f-je

$\left(0, -\frac{1}{25}\right)$ je presjek sa Y-osom



| x | $(-\infty, 1)$ | $(1, 5)$ | $(5 + \infty)$ |
|---|---|---|---|
| Y | − | + | + |

znak f-je

PARNOST (NEPARNOST), PERIODIČNOST

$\mathcal{D}$ nije simetrično $\Rightarrow$

$\Rightarrow$ f-ja nije ni parna ni neparna

PONAŠANJE NA KRAJEVIMA INTERVALA DEFINISANOSTI I ASIMPTOTE

vertikalna asimptota F-ja ima prekid za $x=5$

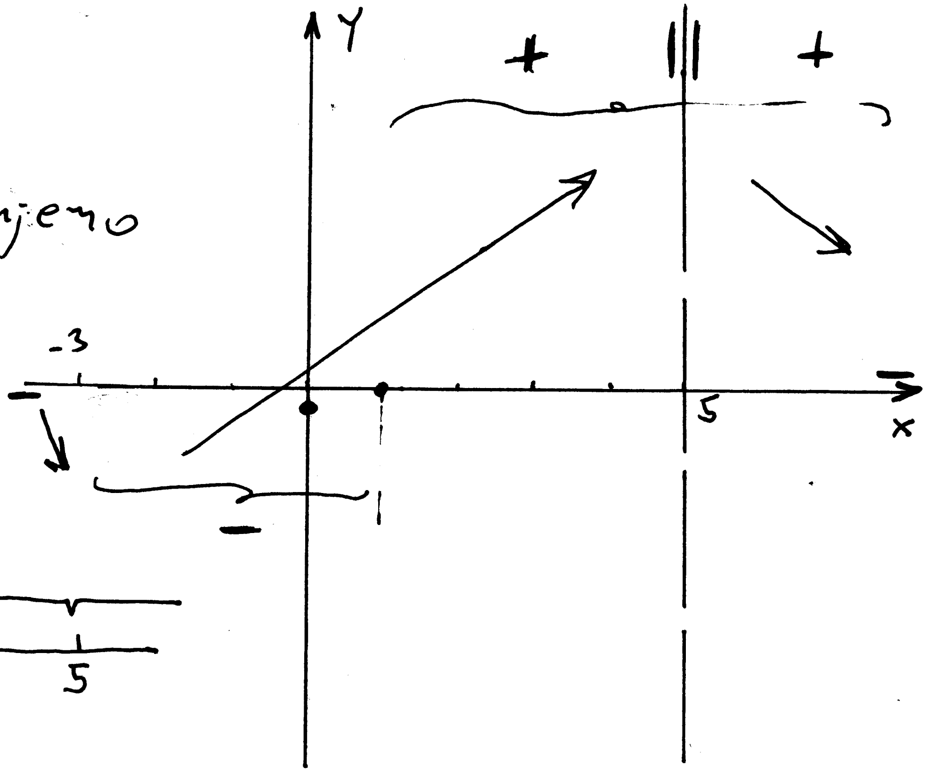
$$\left.\begin{array}{l}\lim\limits_{x\to 5-0} f(x) = \lim\limits_{x\to 5-0} \dfrac{x-1}{(x-5)^2} = \dfrac{5-0-1}{+0} = +\infty \\ \lim\limits_{x\to 5+0} f(x) = \lim\limits_{x\to 5+0} \dfrac{x-1}{(x-5)^2} = \dfrac{5+0-1}{+0} = +\infty\end{array}\right\} \Rightarrow x=5 \text{ je V. A.}$$

horizontalna asimptota

$$\lim\limits_{x\to \mp\infty} f(x) = \lim\limits_{x\to \mp\infty} \dfrac{x-1}{x^2-10x+25} \begin{array}{l} /:x \\ /:x \end{array} = 0 \Rightarrow Y=0 \text{ je H. A.}$$

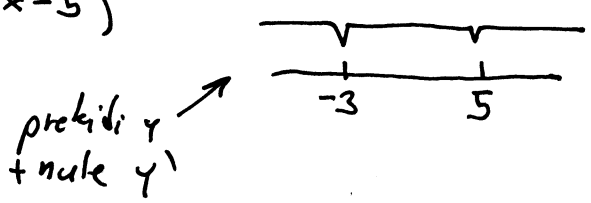
f-ja nema kosu asimptotu

Poslije ovog koraka počinjemo skicirati graf f-je.



RAST I OPADANJE

$$Y' = -\dfrac{x+3}{(x-5)^3}$$



| x | $(-\infty, -3)$ | $(-3, 5)$ | $(5, +\infty)$ |
|---|---|---|---|
| $y'$ | $-$ | $+$ | $-$ |
| $y$ | ↘ | ↗ | ↘ |

tabela rasta i opadanja

MIN

$$f(-3) = -\frac{1}{16}$$

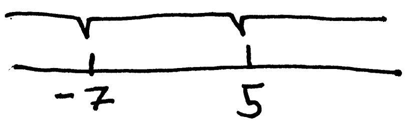
EKSTREMI F-JE

Na osnovu tabele rasta i opadanja f-ja ima minimum u tački $M(-3, -\frac{1}{16})$

PREVOJNE TAČKE I INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$$y'' = \frac{2(x+7)}{(x-5)^4}$$

prekidi y + nule y''



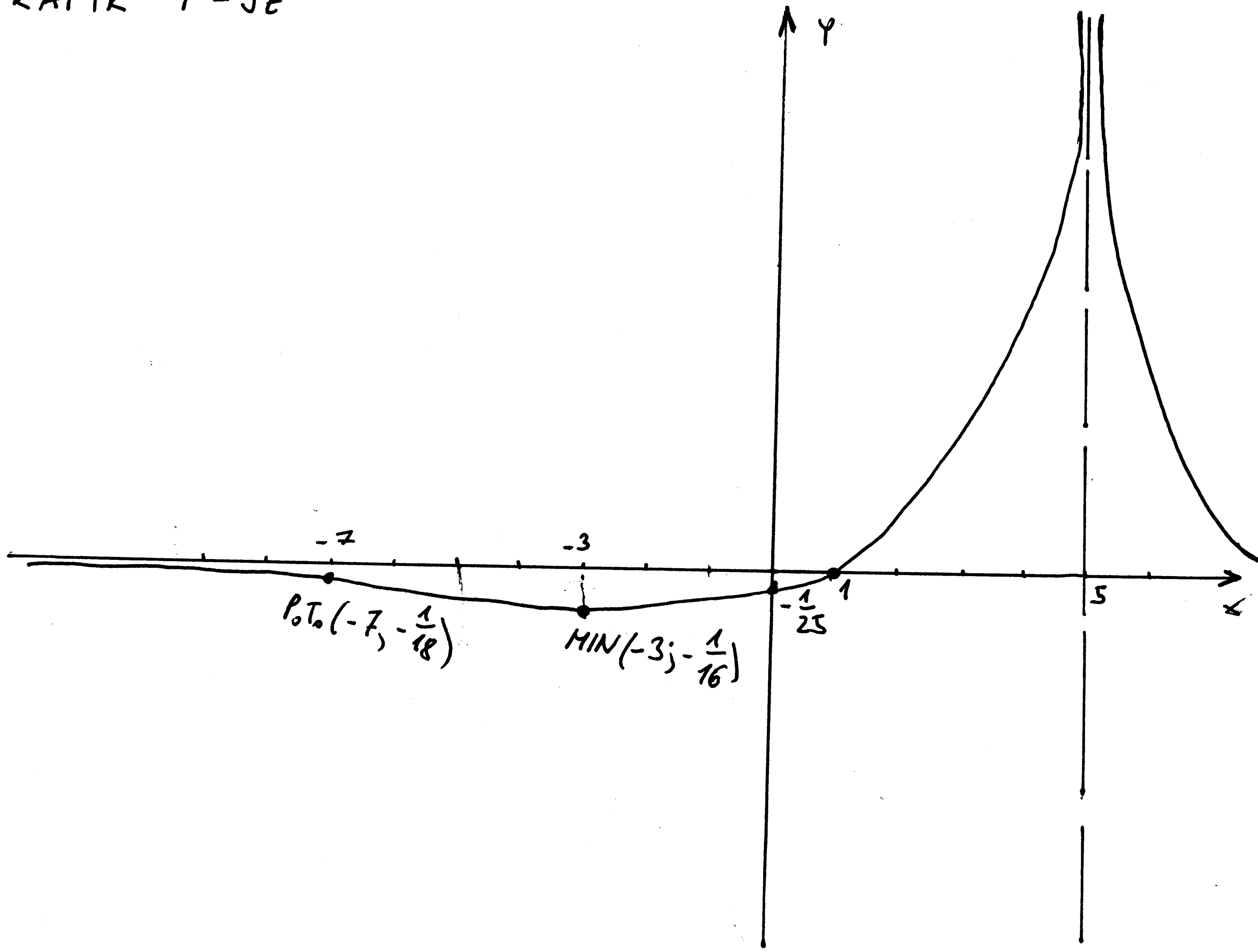
| x | $(-\infty, -7)$ | $(-7, 5)$ | $(5, +\infty)$ |
|---|---|---|---|
| $y''$ | $-$ | $+$ | $+$ |
| $y$ | $\cap$ | $\cup$ | $\cup$ |

tabela konveks. i konkavn.

$P_oT_o$

$P_oT_o(-7, -\frac{1}{18})$

GRAFIK F-JE



(#) Ispitati f-ju i nacrtati njen grafik: $y = \frac{x^3-2}{2x^2}$.

Rj. definiciono područje

$D: x \neq 0$

parnost (neparnost), periodičnost

$f(-x) = \frac{(-x)^3-2}{2(-x)^2} = \frac{-x^3-2}{2x^2} \neq \pm f(x)$

· f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak

$y=0$ akko $x^3-2=0$

$x = \sqrt[3]{2} \approx 1,26$

$(\sqrt[3]{2}, 0)$ je nula f-je

$f(0) =$ nije definisano

f-ja ne siječe y-osu

$2x^2 > 0 \quad \forall x \in D$

$$\left.\begin{array}{ll} y>0 & \text{za} \quad x > \sqrt[3]{2} \\ y<0 & \text{za} \quad x < \sqrt[3]{2} \end{array}\right\} \text{znak f-je.}$$

ponašanje na krajevima intervala definisanosti i asimptote

za $x=0$ f-ja ima prekid

$$\left.\begin{array}{l} \lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0-} \frac{x^3-2}{2x^2} = \frac{(0-)^3-2}{2(0-)^2} = \frac{-2-0}{0+} = -\infty \\ \lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0+} \frac{(0+)^3-2}{2(0+)^2} = \frac{-2+0}{+0} = -\infty \end{array}\right\} \Rightarrow \begin{array}{l} x=0 \\ \text{je V. A.} \end{array}$$

$\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \frac{x^3-2 \;\; /:x^2}{2x^2 \;\; /:x^2} = \pm\infty$ f-ja nema H. A.

Tražimo kosu asimptotu u obliku $y = kx + n$.

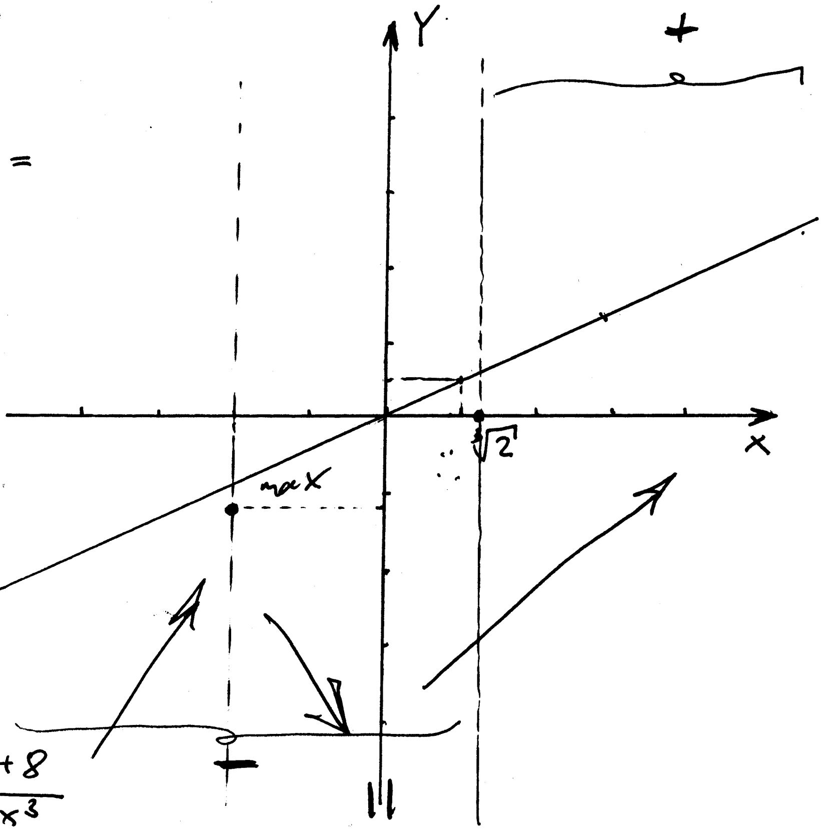
$k = \lim\limits_{x\to\infty} \frac{f(x)}{x} = \lim\limits_{x\to\infty} \frac{x^3-2 \;\; /:x^3}{2x^3 \;\; /:x^3} = \frac{1}{2}$

$n = \lim\limits_{x\to\infty} [f(x) - kx] = \lim\limits_{x\to\infty} \left[\frac{x^3-2}{2x^2} - \frac{1}{2}x\right] =$

$= \lim\limits_{x\to\infty} \frac{x^3-2-x^3}{2x^2} = \lim\limits_{x\to\infty} \frac{-2}{2x^2} = 0$

kosa asimptota je $y = \frac{1}{2}x$

Poslije ovog koraka počinjemo skicirati grafik.



rast i opadanje

$y' = \left(\frac{x^3-2}{2x^2}\right)' = \frac{3x^2 \cdot 2x^2 - (x^3-2)4x}{4x^4} =$

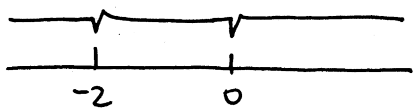
$= \frac{6x^4 - 4x^4 + 8x}{4x^4} = \frac{2x^4+8x}{4x^4} = \frac{x^3+8}{2x^3}$

$y' = \frac{x^3+8}{2x^3}$ , $y'=0$ akko $x^3+8=0$

$x^3=-8$

$x=-2$

prekidi y + nule y'



| x | $(-\infty,-2)$ | $(-2,0)$ | $(0,+\infty)$ |
|---|---|---|---|
| y' | + | − | + |
| y | ↗ | ↘ | ↗ |

max N.D.

$f(-2) = \frac{(-2)^3-2}{2(-2)^2} = \frac{-10}{8} = -\frac{5}{4} \approx -1{,}25$

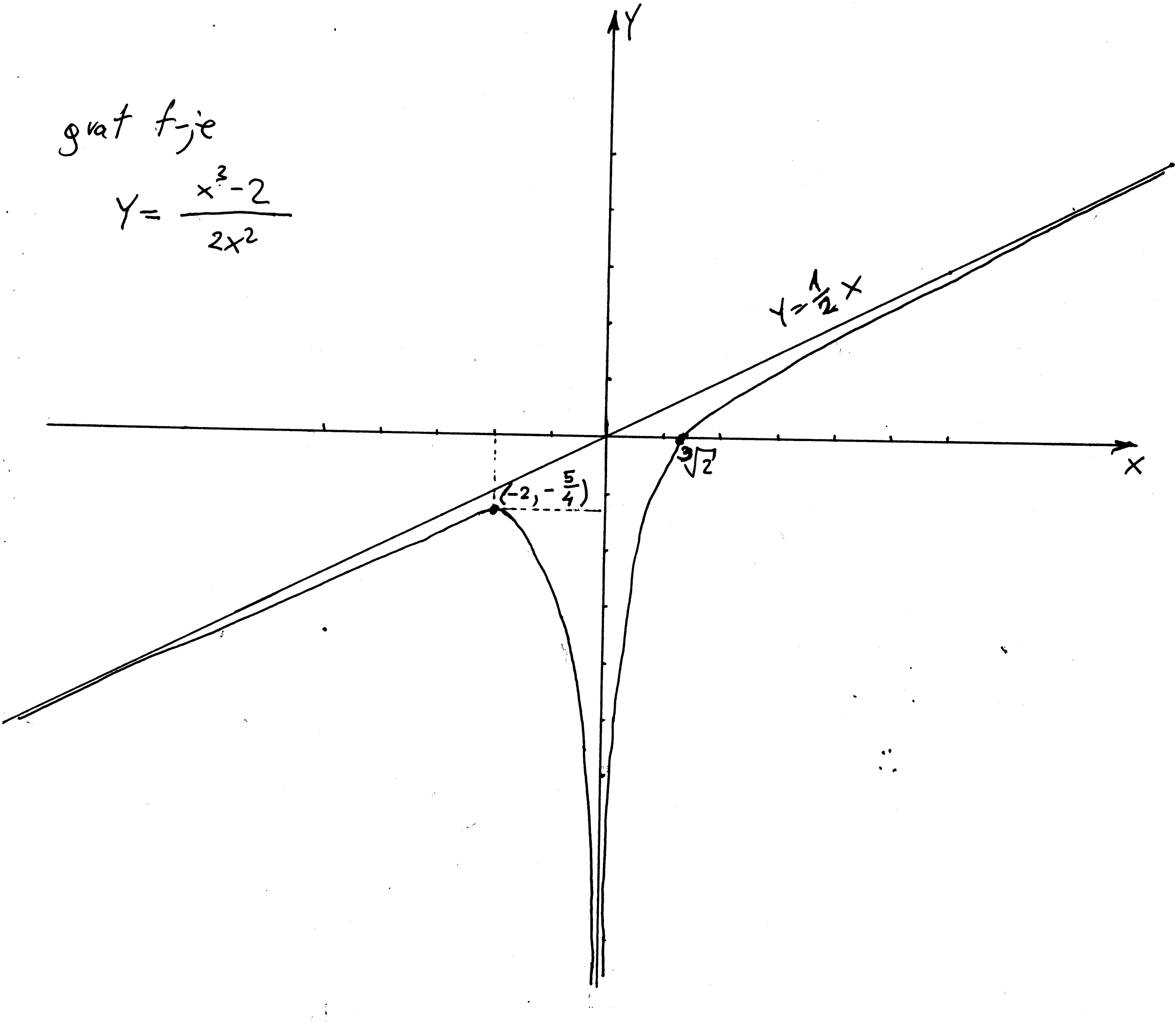
prevojne tacke i intervali konveksnosti i konkavnosti

$$y'' = \left(\frac{x^3+8}{2x^3}\right)' = \frac{3x^2\cdot 2x^3-(x^3+8)\cdot 6x^2}{4x^6} = \frac{6x^5-6x^5-48}{4x^6} = \frac{-48}{4x^6} = -\frac{12}{x^6} < 0$$

F-ja nema prevojnih tački i uvijek je negativna sto znači uvijek je $\cap$ oblika.

graf f-je

$y = \frac{x^3-2}{2x^2}$



(#) Ispitati f-ju i nacrtati njen grafik $y=\frac{x^2+10}{x^2+4x+4}$.

Rj. $y=\frac{x^2+10}{x^2+4x+4}=\frac{x^2+10}{(x+2)^2}$

definiciono područje

$x+2\neq 0$ $\quad D: x\in(-\infty,-2)\cup(-2,+\infty)$

$x\neq -2$

·parnost (neparnost), periodičnost

$D$ nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa $y$-osom i znak f-je

$y=0 \Rightarrow x^2+10=0$

Kako je $x^2+10>0\ \forall x\in D$ to f-ja nema nule

$f(0)=\frac{0+10}{(0+2)^2}=\frac{10}{4}=\frac{5}{2}$

$(0,\frac{5}{2})$ je presjek sa $y$-osom

$x^2+10>0\ \forall x\in D$ $\quad$ f-ja je

$(x+2)^2>0\ \forall x\in D$ $\quad$ uvijek pozitivna

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-2$ f-ja ima prekid

$\lim\limits_{x\to-2-0} f(x)=\lim\limits_{x\to-2-0}\frac{x^2+10}{(x+2)^2}=\frac{(-2-0)^2+10}{(-2-0+2)^2}=\frac{14+0}{+0}=+\infty \Rightarrow x=-2$ je V. A. (sa lijeve strane)

$\lim\limits_{x\to-2+0} f(x)=\lim\limits_{x\to-2+0}\frac{x^2+10}{(x+2)^2}=\frac{(-2+0)^2+10}{(-2+0+2)^2}=\frac{14-0}{+0}=+\infty \Rightarrow x=-2$ je V. A. (sa desne strane)

$\lim\limits_{x\to\mp\infty} f(x)=\lim\limits_{x\to\mp\infty}\frac{x^2+10\ :x^2}{x^2+4x+4:x^2}=\lim\limits_{x\to\mp\infty}\frac{1+\overset{\nearrow 0}{\frac{10}{x^2}}}{1+\underset{\swarrow 0}{\frac{4}{x}}+\underset{\swarrow 0}{\frac{4}{x^2}}}=1 \Rightarrow y=1$ je H. A.

f-ja nema kosu asimptotu

Poslije ovog koraka počinjemo skicirati grafik.

rast i opadanje

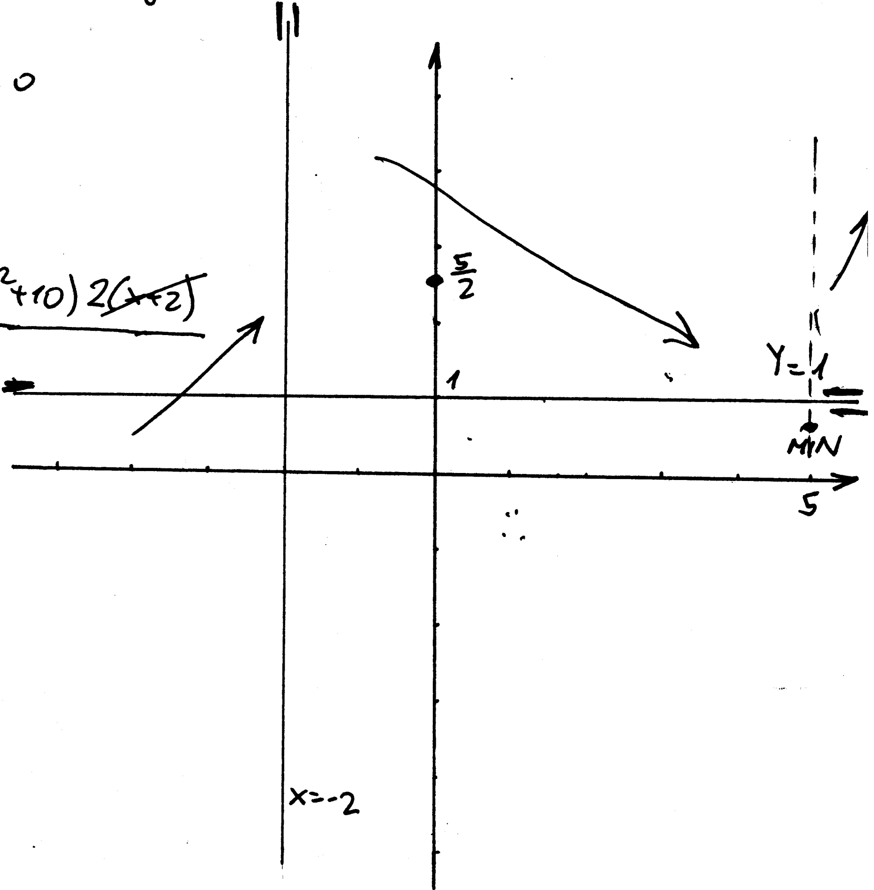
$$y'=\left(\frac{x^2+10}{(x+2)^2}\right)'=\frac{2x\cdot(x+2)^{\cancel{2}}-(x^2+10)2\cancel{(x+2)}}{(x+2)^{\cancel{4}3}}$$

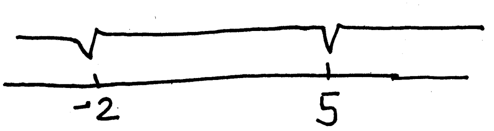
$$y'=\frac{\cancel{2x^2}+4x-\cancel{2x^2}-20}{(x+2)^3}$$

$$y'=\frac{4x-20}{(x+2)^3}=4\frac{x-5}{(x+2)^3}$$

$y'=0$ akko $x-5=0$

$x=5$





← prekidi Y + nule Y'

| x | $(-\infty,-2)$ | $(-2,5)$ | $(5,+\infty)$ |
|---|---|---|---|
| Y' | + | − | + |
| Y | ↗ | ↘ | ↗ |

min

rast i opadanje

ekstremi f-je

Stacionarna tačka je $x=5$.
Na osnovu tabele rasta i opadanja vidimo da f-ja u toj tački ima ekstrem i to minimum

$f(5)=\frac{25+10}{7^2}=\frac{35}{49}\approx 0,71$ $\left(5,\frac{35}{49}\right)$ je tačka minimuma

prevojne tačke i intervali konveksnosti i konkavnosti

$$Y''=\left(4\frac{x-5}{(x+2)^3}\right)'=4\frac{1\cdot(x+2)^{\cancel{3}}-(x-5)3\cancel{(x+2)^2}}{(x+2)^{\cancel{6}4}}=4\frac{x+2-3x+15}{(x+2)^4}.$$

$$Y''=4\frac{-2x+17}{(x+2)^4}=-4\frac{2x-17}{(x+2)^4}$$

← nule Y'' + prekidi Y

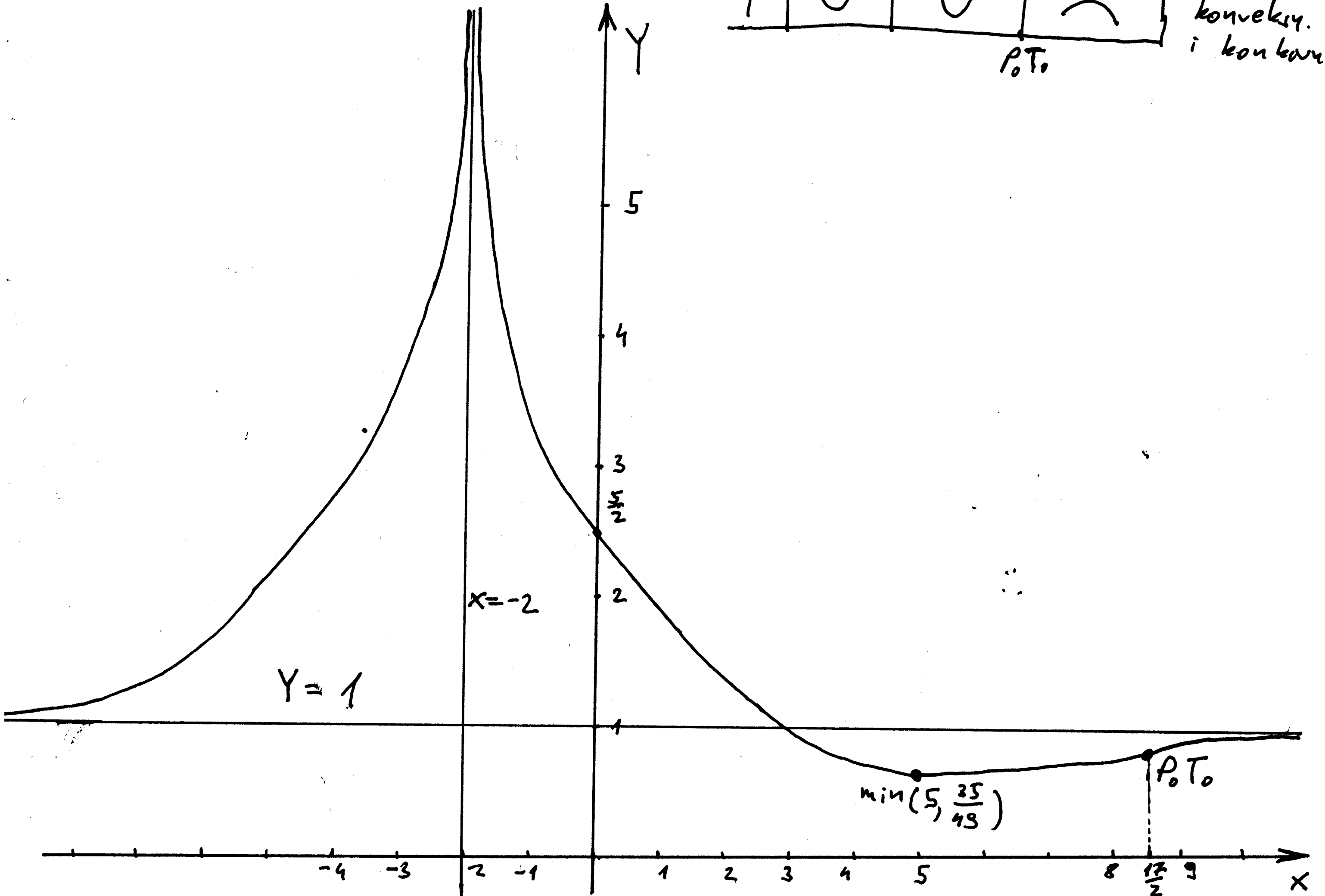
$Y''=0$ akko $2x-17=0$

$x=\frac{17}{2}$

| x | $(-\infty,-2)$ | $(-2,\frac{17}{2})$ | $(\frac{17}{2},+\infty)$ |
|---|---|---|---|
| Y'' | + | + | − |
| Y | ∪ | ∪ | ∩ |

$P_oT_o$

intervali konveksn. i konkavn.



④ Odrediti parametre a i b tako da f-ja

a) $Y=\frac{(ax+b)^4}{x^3}$ ima kosu asimptotu u pravoj $Y=x-4$

b) $Y=\frac{(ax+b)^3}{x^2}$ ima kosu asimptotu u pravoj $Y=27x+9$

c) $Y=\frac{(ax+b)^2}{x}$ ima kosu asimptotu u pravoj $Y=4x+4$

d) $Y=\frac{a^2x^3+b^3x^2+1}{x^2}$ ima kosu asimptotu u pravoj $Y=64x-27$

Rj.-upute

Kosa asimptota je u obliku $y=kx+n$ gdje je $k=\lim\limits_{x\to\infty}\frac{f(x)}{x}$, $n=\lim\limits_{x\to\infty}(f(x)-kx)$

a) $k=\lim\limits_{x\to\infty}\frac{f(x)}{x}=\lim\limits_{x\to\infty}\frac{(ax+b)^4}{x^4}=\lim\limits_{x\to\infty}\frac{a^4x^4+\dots}{x^4}=a^4$ $\quad a^4=1$

$a=1$

$$\begin{array}{c}1\\1\;1\\1\;2\;1\\1\;3\;3\;1\\1\;4\;6\;4\;1\end{array}$$

$n=\lim\limits_{x\to\infty}[f(x)-kx]=\lim\limits_{x\to\infty}\left[\frac{(x+b)^4}{x^3}-x\right]=\lim\limits_{x\to\infty}\frac{x^4+4bx^3+\dots-x^4}{x^3}=4b$

$4b=-4 \Rightarrow b=-1$ Traženi parametri su $a=1$, $b=-1$

b) $k=\lim\limits_{x\to\infty}\frac{(ax+b)^3}{x^3}=\lim\limits_{x\to\infty}\frac{a^3x^3+\dots}{x^3}=a^3 \Rightarrow a^3=27 \Rightarrow a=\sqrt[3]{27}=3$

$n=\lim\limits_{x\to\infty}\left[\frac{(3x+b)^3}{x^2}-27x\right]=\lim\limits_{x\to\infty}\frac{27x^3+9x^2b+\dots-27x^3}{x^2}=9b \Rightarrow$ $9b=9$

$b=1$

c) $k=\lim\limits_{x\to\infty}\frac{(ax+b)^2}{x^2}=\lim\limits_{x\to\infty}\frac{a^2x^2+2abx+b^2}{x^2}=a^2 \Rightarrow a^2=4 \Rightarrow a=2$

$n=\lim\limits_{x\to\infty}\left[\frac{(2x+b)^2}{x}-4x\right]=\lim\limits_{x\to\infty}\frac{4x^2+4bx+b^2-4x^2}{x}=4b \Rightarrow$ $4b=16$

$b=4$

d) $k=\lim\limits_{x\to\infty}\frac{a^2x^3+b^3x^2+1}{x^3}=a^2 \Rightarrow a^2=64 \Rightarrow a=8$

$b=-3$

$n=\lim\limits_{x\to\infty}\left[\frac{64x^3+b^3x^2+1}{x^2}-64x\right]=\lim\limits_{x\to\infty}\frac{b^3x^2+1}{x^2}=b^3 \Rightarrow b^3=-27$

# (#) Odrediti definiciono područje, ekstreme, prevojne tačke, te intervale konveksnosti i konkavnosti f-je

$$y = \frac{3x^2 - 15x + 108}{x-5}$$

R.-upute:

DEFINICIONO PODRUČJE

$$x - 5 \neq 0$$
$$x \neq 5$$

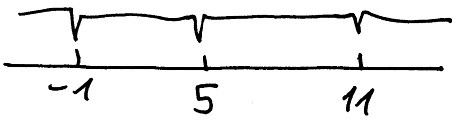
$$D: x \in (-\infty, 5) \cup (5, +\infty)$$
$$x \in R \setminus \{5\}$$

EKSTREMI F-JE

Na osnovu tabele rasta i opadanja vidimo da f-ja ima dva ekstrema i to $MAX(-1; -21)$ i $MIN(11; 51)$

INTERVALI RASTA I OPADANJA

$$y' = \frac{3(x+1)(x-11)}{(x-5)^2} = \frac{3x^2 - 30x - 33}{(x-5)^2}$$



← prekidi y + nule y'

| x | $(-\infty, -1)$ | $(-1, 5)$ | $(5, 11)$ | $(11, +\infty)$ |
|---|---|---|---|---|
| y' | + | − | − | + |
| y | ↗ | ↘ | ↘ | ↗ |
| | MAX | | MIN | |

tabela rasta i opadanja

$f(-1) = -21$; $f(11) = 51$

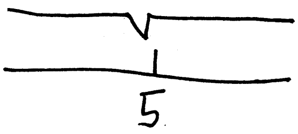
PREVOJNE TAČKE I INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$$y'' = \frac{216}{(x-5)^3}$$

$y''(-1) = -1 < 0$

$y''(11) = 1 > 0$

$y'' \neq 0 \; \forall x \in D \Rightarrow$ f-ja nema prevojnih tački



← prekidi y + nule y''

| x | $(-\infty, 5)$ | $(5, +\infty)$ |
|---|---|---|
| y'' | − | + |
| y | ∩ | ∪ |

tabela konveksnosti i konkavnosti

(#) Odrediti definiciono područje, ekstreme, prevojne tačke, te intervale konveksnosti i konkavnosti f-je

$$y = \frac{2x^2 - 6x + 2}{x - 3}$$

Rj.-upute:

DEFINICIONO PODRUČJE

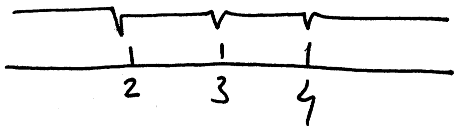
$x - 3 \neq 0$

$x \neq 3$

$\mathcal{D} : x \in (-\infty, 3) \cup (3, +\infty)$

INTERVALI RASTA I OPADANJA

$$y' = \frac{2(x-2)(x-4)}{(x-3)^2} = \frac{2x^2 - 12x + 16}{(x-3)^2}$$



← prekidi y + nule y'

| x | $(-\infty, 2)$ | $(2, 3)$ | $(3, 4)$ | $(4, +\infty)$ |
|---|---|---|---|---|
| y' | + | − | − | + |
| Y | ↗ | ↘ | ↘ | ↗ |

MAX MIN

tabela rasta i opadanja

$$f(2) = \frac{8 - 12 + 2}{-1} = 2$$

$f(4) = 10$

EKSTREMI F-JE

Na osnovu tabele rasta i opadanja vidimo da f-ja ima dva ekstrema i to MAX$(2; 2)$ i MIN$(4; 10)$

PREVOJNE TAČKE I INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$$y'' = \frac{4}{(x-3)^3}$$

$y'' \neq 0 \quad \forall x \in \mathcal{D} \Rightarrow$ f-ja nema prevojnih tački

$y''(2) = -4 < 0$

$y''(4) = 4 > 0$

← prekidi y + nule y''

| x | $(-\infty, 3)$ | $(3, +\infty)$ |
|---|---|---|
| y'' | − | + |
| Y | ∩ | ∪ |

tabela konveksnosti i konkavnosti

(#) Odrediti definiciono područje, ekstreme, prevojne tačke, te intervale konveksnosti i konkavnosti f-je

$$y=\frac{4x^2+8x+1}{x+2}$$

Rj.-upute:

DEFINICIONO PODRUČJE

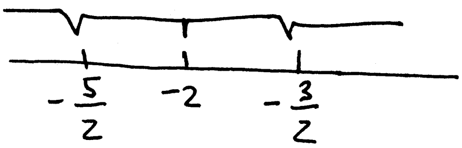
$x+2\neq 0$

$x\neq -2$

$\mathcal{D}: x\in(-\infty,-2)\cup(-2,+\infty)$

INTERVALI RASTA I OPADANJA

$$y'=\frac{(2x+5)(2x+3)}{(x+2)^2}=\frac{4x^2+16x+15}{(x+2)^2}$$



← prekidi y + nule y'

| x | $(-\infty,-\frac{5}{2})$ | $(-\frac{5}{2},-2)$ | $(-2,-\frac{3}{2})$ | $(-\frac{3}{2},+\infty)$ |
|---|---|---|---|---|
| y' | + | — | — | + |
| y | ↗ | ↘ | ↘ | ↗ |

MAX

tabela rasta i opadanja

MIN

EKSTREMI F-JE

Na osnovu tabele rasta i opadanja vidimo da f-ja ima dva ekstrema i to

$MAX(-\frac{5}{2};-12)$ i $MIN(-\frac{3}{4};-4)$

$f(-\frac{5}{2})=-12$; $f(-\frac{3}{2})=-4$

PREVOJNE TAČKE I INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$$y''=\frac{2}{(x+2)^3}$$

$y''\neq 0 \ \forall x\in\mathcal{D} \Rightarrow$ f-ja nema prevojnih tački

← prekidi y + nule y''

$y''(-\frac{5}{2})=-16<0$

$y''(-\frac{3}{2})=16>0$

| x | $(-\infty,-2)$ | $(-2,+\infty)$ |
|---|---|---|
| y'' | — | + |
| y | ∩ | ∪ |

tabela konveksnosti i konkavnosti

(#) Ispitati f-ju i nacrtati njen grafik $y=\frac{3x^3-1}{(x+1)^3}$..

Rj.-upute

DEFINICIONO PODRUČJE

$D: x\in \mathbb{R}\setminus\{-1\}$

$x\in(-\infty,-1)\cup(-1,+\infty)$

PARNOST (NEPARNOST), PERIODIČNOST

D nije simetrično pa f-ja nije ni parna ni neparna

F-ja nije periodična.

NULE, PRESJEK SA y-OSOM, ZNAK

Nula f-je je $(\frac{1}{\sqrt[3]{3}};0)$.

Presjek sa y-osom je $(0;-1)$

| x | $(-\infty,-1)$ | $(-1,\frac{1}{\sqrt[3]{3}})$ | $(\frac{1}{\sqrt[3]{3}},+\infty)$ |
|---|---|---|---|
| y | + | − | + |

znak f-je

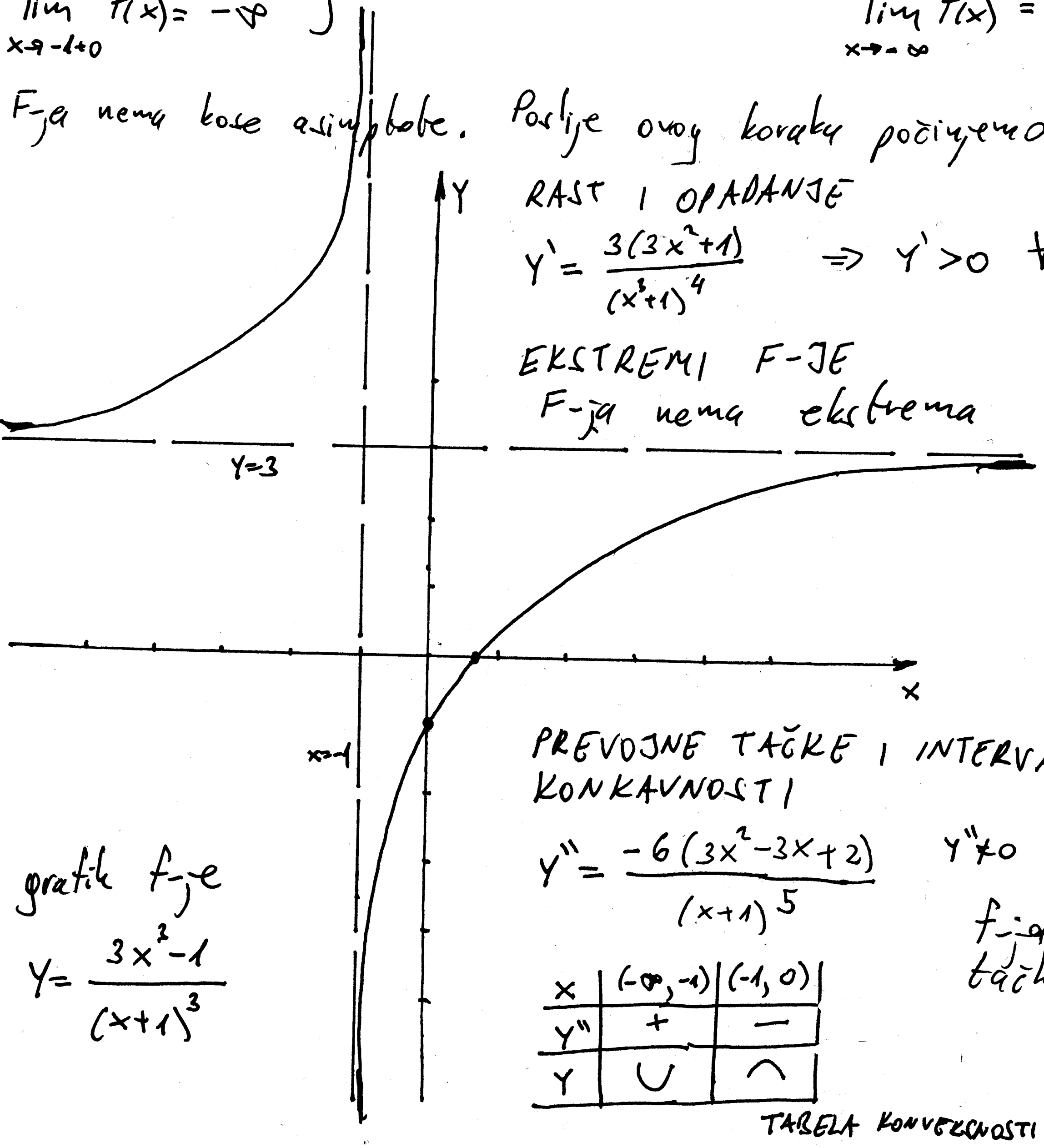
PONAŠANJE NA KRAJEVIMA INTERVALA DEFINISANOSTI I ASIMPTOTE

F-ja ima prekid za $x=-1$.

$$\left.\begin{array}{l}\lim\limits_{x\to-1-0} f(x)=+\infty\\ \lim\limits_{x\to-1+0} f(x)=-\infty\end{array}\right\}\Rightarrow x=-1 \text{ je V.A.}$$

$$\left.\begin{array}{l}\lim\limits_{x\to+\infty} f(x)=3\\ \lim\limits_{x\to-\infty} f(x)=3\end{array}\right\}\Rightarrow y=3 \text{ je H.A.}$$

F-ja nema kose asimptote. Poslije ovog koraka počinjemo skicirati graf f-je.



RAST I OPADANJE

$y'=\frac{3(3x^2+1)}{(x^3+1)^4}$ $\Rightarrow y'>0\ \forall x$ f-ja uvijek raste

EKSTREMI F-JE

F-ja nema ekstrema

PREVOJNE TAČKE I INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$y''=\frac{-6(3x^2-3x+2)}{(x+1)^5}$ $y''\neq 0$ za $\forall x\in D$

f-ja nema prevojnih tački

| x | $(-\infty,-1)$ | $(-1,0)$ |
|---|---|---|
| y'' | + | − |
| y | ∪ | ∩ |

TABELA KONVEKSNOSTI I KONKAVNOSTI

grafik f-je

$y=\frac{3x^3-1}{(x+1)^3}$

(#) Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $y=x^2e^{-\frac{x}{3}}$.

Rj:

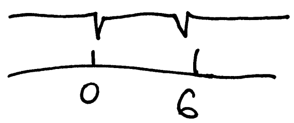
definiciono područje
$x \in R$

| x | $(-\infty, 0)$ | $(0, 6)$ | $(6, +\infty)$ |
|---|---|---|---|
| $y'$ | $-$ | $+$ | $-$ |
| $y$ | ↘ | ↗ | ↘ |
|  |  | MIN | MAX |

rast i opadanje

$$y' = -\frac{1}{3}x(x-6)e^{-\frac{x}{3}}$$

$y'=0$ akko $x=0$ ili $x=6$



prekidi y + nule y'

$f(0)=0$, $f(6)=36e^{-2}=\frac{36}{e^2}$

F-ja ima ekstreme, i to MIN(0,0), MAX$(6, \frac{36}{e^2})$.

$$y'' = \frac{1}{9}(x^2-12x+18)e^{-\frac{x}{3}}$$

| x | $(-\infty, 6-3\sqrt{2})$ | $(6-3\sqrt{2}, 6+3\sqrt{2})$ | $(6+3\sqrt{2}, +$ |
|---|---|---|---|
| $y''$ | $+$ | $-$ | $+$ |
| $y$ | $\cup$ | $\cap$ | $\cup$ |
|  |  | P.T. | P.T. |

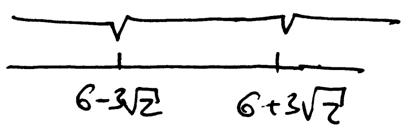
intervali konveksnosti i konkavnosti

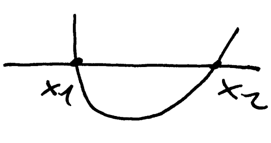
$y''=0$ akko $x^2-12x+18=0$

$D=144-72=72=2\cdot4\cdot9$

$x_{1,2}=\frac{12\pm6\sqrt{2}}{2}$, $x_1=6-3\sqrt{2}$, $x_2=6+3\sqrt{2}$

prekidi y + nule y''





$f(6-3\sqrt{2})=(6-3\sqrt{2})^2e^{-2+\sqrt{2}}$

$f(6+3\sqrt{2})=(6+3\sqrt{2})^2e^{-2-\sqrt{2}}$

Prevojne tačke su $P_1(6-3\sqrt{2}, (6-3\sqrt{2})^2e^{-2+\sqrt{2}})$ i

$P_2(6+3\sqrt{2}, (6+3\sqrt{2})^2e^{-2-\sqrt{2}})$.

# Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $y = xe^{-\frac{1}{x}}$.

Rj:

definiciono područje

$x \neq 0$

$x \in \mathbb{R} \setminus \{0\}$

$x \in (-\infty, 0) \cup (0, +\infty)$

| x | $(-\infty,-1)$ | $(-1,0)$ | $(0,+\infty)$ |
|---|---|---|---|
| $y'$ | + | − | + |
| $y$ | ↗ | ↘ | ↗ |

tabela rasta i opadanja

MAX

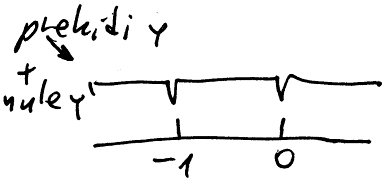
$$y' = \frac{(x+1)\,e^{-\frac{1}{x}}}{x}$$

$$f(-1) = (-1)e^{1} = -e$$

$y' = 0$ akko $x+1=0$

$x = -1$

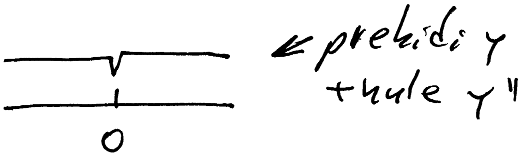
F-ja ima maksimum u tački $M(-1,-e)$



$$y'' = \frac{e^{-\frac{1}{x}}}{x^3}$$

F-ja nema prevojnih tački

$y'' \neq 0 \ \forall x \in \mathcal{D}$



| x | $(-\infty,0)$ | $(0,+\infty)$ |
|---|---|---|
| $y''$ | − | + |
| $y$ | ∩ | ∪ |

tabela konveksnosti i konkavnosti

# ⧣ Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $y = x \cdot e^{-\frac{x^2}{4}}$.

Rj.

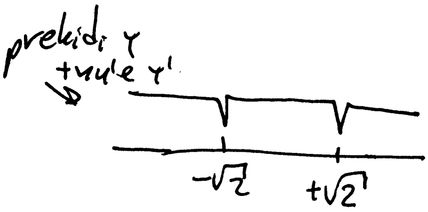
definiciono područje

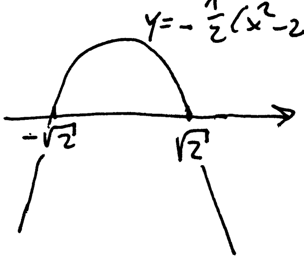
$x \in \mathbb{R}$

$$y' = -\frac{1}{2}(x^2-2)\, e^{-\frac{x^2}{4}}$$

$y' = 0$ akko $x^2 - 2 = 0$

$x_{1,2} = \pm\sqrt{2}$





| x | $(-\infty, -\sqrt{2})$ | $(-\sqrt{2}, +\sqrt{2})$ | $(\sqrt{2}, +\infty)$ |
|---|---|---|---|
| y' | − | + | − |
| y | ↘ | ↗ | ↘ |

MIN MAX

tabela rasta i opadanja

$f(-\sqrt{2}) = -\sqrt{2}\, e^{-\frac{1}{2}}$

$f(\sqrt{2}) = \sqrt{2}\, e^{-\frac{1}{2}}$

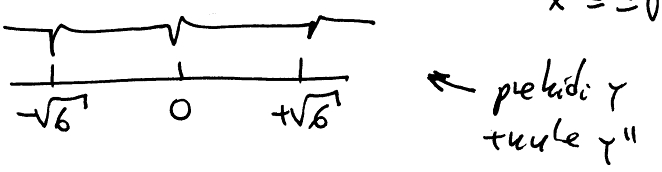
Ekstremi f-je su $MIN\left(-\sqrt{2}, -\frac{\sqrt{2}}{\sqrt{e}}\right)$ i $MAX\left(\sqrt{2}, \frac{\sqrt{2}}{\sqrt{e}}\right)$.

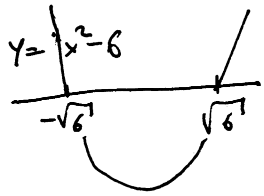
$$y'' = \frac{1}{4}\, x(x^2-6)\, e^{-\frac{x^2}{4}}$$

$y'' = 0$ akko $x(x^2-6) = 0$

$x = 0$ ili $x^2 = 6$

$x = \pm\sqrt{6}$





| x | $(-\infty, -\sqrt{6})$ | $(-\sqrt{6}, 0)$ | $(0, \sqrt{6})$ | $(\sqrt{6}, +\infty)$ |
|---|---|---|---|---|
| y'' | − | + | − | + |
| y | ∩ | ∪ | ∩ | ∪ |

P.T. P.T. P.T. tabela

$f(-\sqrt{6}) = -\sqrt{6}\, e^{-\frac{6}{4}}$

$f(0) = 0$

$f(\sqrt{6}) = \sqrt{6}\, e^{-\frac{6}{4}}$

Prevojne tačke su $P_1(-\sqrt{6}, -\sqrt{6}\, e^{-\frac{3}{2}})$, $P_2(0,0)$ i $P_3(\sqrt{6}, \sqrt{6}\, e^{-\frac{3}{2}})$.

(#) Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $y = x e^{-\frac{x}{2}}$.

Rj: definiciono područje

$x \in \mathbb{R}$

$x \in (-\infty, +\infty)$

| x | $(-\infty, 2)$ | $(2, +\infty)$ |
|---|---|---|
| $y'$ | + | − |
| y | ↗ | ↘ |

MAX

tabela rasta i opadanja

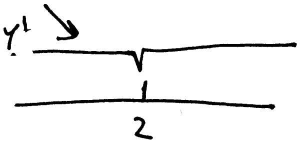
$$y' = -\frac{1}{2}(x-2)\, e^{-\frac{x}{2}}$$

$y'=0$ akko $x-2=0$

$x=2$

$$f(2) = 2 e^{-\frac{(2)}{2}} = 2e^{-1}$$

prekidi y + nule y'



F-ja ima ekstrem i to maksimum u tački M(2, 2e).

$$y'' = \frac{1}{4}(x-4)\, e^{-\frac{x}{2}}$$

$y''=0$ akko $(x-4)=0$

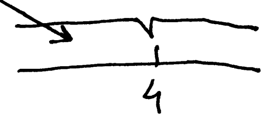
$x=4$

| x | $(-\infty, 4)$ | $(4, +\infty)$ |
|---|---|---|
| $y''$ | − | + |
| y | ∩ | ∪ |

P.T.

tabela konveksnosti i konkavnosti

prekidi y + nule y'



$$f(4) = 4e^{-\frac{4}{2}}$$

F-ja ima prevojnu tačku $P(4, 4e^{-2})$

(#) Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $y = \frac{e^{2x}}{x+1}$.

Rj.-upute

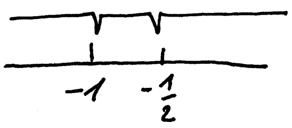
D.p. $x+1 \neq 0$

$x \in (-\infty,-1) \cup (-1,+\infty)$

$y' = \frac{e^{2x}(2x+1)}{(x+1)^2}$

$y' = 0$ akko $2x+1=0$

$x = -\frac{1}{2}$



← prekidi f-je Y
+ nule f-je Y'

| x | $(-\infty,-1)$ | $(-1,-\frac{1}{2})$ | $(-\frac{1}{2},+\infty)$ |
|---|---|---|---|
| Y' | − | − | + |
| Y | ↘ | ↘ | ↗ |
|  |  | MIN |  |

tabela rasta i opadanja

$f(-\frac{1}{2}) = \frac{e^{-1}}{\frac{1}{2}} = \frac{2}{e}$

$MIN(-\frac{1}{2}, \frac{2}{e})$

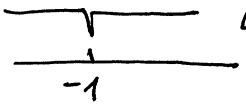
$y'' = \frac{2e^{2x}(2x^2+2x+1)}{(x+1)^3}$

$2x^2+2x+1=0$

$D = 4-8<0$

$2x^2+2x+1>0 \quad \forall x$

$y'' \neq 0 \quad \forall x \in R$



← prekidi Y
+ nule Y''

| x | $(-\infty,-1)$ | $(-1,+\infty)$ |
|---|---|---|
| Y'' | − | + |
| Y | ⌢ | ⌣ |

tabela konveksnosti i konkavnosti

F-ja nema prevojnih tački.

(#) Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $y=\frac{e^{3x}}{1+e^{-x}}$.

Rj. -upute:

$1+e^{-x}\neq 0$   D.p. $x\in R$

$\underbrace{e^{-x}}_{>0\ \forall x}\neq -1$   $x\in(-\infty,+\infty)$

$y'=0$ akko $4e^{-x}+3=0$

$\underbrace{e^{-x}}_{>0\ \forall x}=\frac{-3}{4}$

$$y'=\frac{e^{3x}(4e^{-x}+3)}{(1+e^{-x})^2}$$

$y'>0$ za $\forall x\in R$

F-ja nema ekstrem i raste za $\forall x$

$$y''=\frac{e^{4x}(9e^{2x}+23e^{x}+16)}{(e^{x}+1)^3}$$

$e^{x}=t$

$9t^2+23t+16=0$

$D=529-576<0$

$y''\neq 0$ za $\forall x\in R$

| $x$ | $(-\infty,+\infty)$ |
|---|---|
| $y''$ | $+$ |
| $y$ | $\cup$ |

tabela konveksnost i konkavnosti

F-ja nema prevojnih tački.

(#) Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $y = \frac{e^{2x}}{1+e^{2x}}$.

Rj. -upute:

$e^{2x} > 0 \quad \forall x \in R$

D.p. $x \in R$

$x \in (-\infty, +\infty)$

$$y' = \frac{2e^{2x}}{(1+e^{2x})^2}$$

$y' \neq 0 \quad \forall x \in R$

F-ja nema ekstrem

$y' > 0 \quad \forall x \in R$

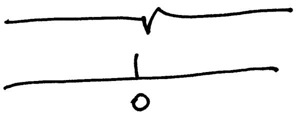
F-ja raste za $\forall x$.

$$y'' = -4\frac{e^{2x}(e^{2x}-1)}{(e^{2x}+1)^3}$$

$y'' = 0$ akko $\underbrace{e^{2x}}_{>0}(e^{2x}-1) = 0$

$e^{2x} = 1$

$x = 0$



← prekidi y + nule y''

| x | $(-\infty, 0)$ | $(0, +\infty)$ |
|---|---|---|
| y'' | + | − |
| y | $\cup$ | $\cap$ |

tabela konveksnosti i konkavnosti

P.T.

$$f(0) = \frac{1}{1+1} = \frac{1}{2}$$

P.T. $\left(0, \frac{1}{2}\right)$

(#) Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $y = \frac{e^{3x}}{1-x}$.

Rj.-upute:

D.p. $1-x \neq 0$

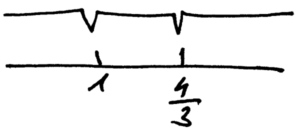
$x \neq 1$

$x \in (-\infty, 1) \cup (1, +\infty)$

$$y' = -\frac{e^{3x}(3x-4)}{(x-1)^2}$$

$y' = 0$ akko $3x-4=0$

$x = \frac{4}{3}$



← prekidi f-je y + nule y'

| x | $(-\infty, 1)$ | $(1, \frac{4}{3})$ | $(\frac{4}{3}, +\infty)$ |
|---|---|---|---|
| y' | + | + | − |
| y | ↗ | ↗ | ↘ |

MAX

tabela rasta i opadanja

$$f\left(\frac{4}{3}\right) = \frac{e^4}{1-\frac{4}{3}} = -3e^4$$

$MAX\left(\frac{4}{3}, -3e^4\right)$

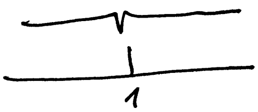
$$y'' = -\frac{e^{3x}(9x^2-24x+17)}{(x-1)^3}$$

$9x^2-24x+17=0$

$D = 576 - 612 < 0$

$9x^2-24x+17 > 0 \quad \forall x$

F-ja nema prevojnih tački



← prekidi y + nule y''

| x | $(-\infty, 1)$ | $(1, +\infty)$ |
|---|---|---|
| y'' | + | − |
| y | ∪ | ∩ |

tabela konveksnosti i konkavnosti

## (#) Ispitati f-ju i nacrtati njen grafik

$$Y=\frac{e^{2x}}{e^{2x}-e^{-x}}$$

Rj. DEFINICIONO PODRUČJE

$\mathcal{D}: x\in(-\infty,0)\cup(0,+\infty)$

$x\in\mathbb{R}\setminus\{0\}$

PARNOST (NEPARNOST), PERIODIČNOST

f-ja nije ni parna ni neparna

f-ja nije periodična

NULE, PRESJEK SA Y-OSOM, ZNAK

f-ja nema nule

F-ja ne siječe Y-osu

| X | $(-\infty,0)$ | $(0,+\infty)$ |
|---|---|---|
| Y | − | + |

znak f-je

RAST I OPADANJE

$$Y'=(-3)\frac{e^{x}}{(e^{2x}-e^{-x})^2}$$

| X | $(-\infty,0)$ | $(0,+\infty)$ |
|---|---|---|
| Y' | − | − |
| Y | ↘ | ↘ |

tabela rasta i opadanja

PONAŠANJE NA KRAJEVIMA INTERVALA DEFINISANOSTI I ASIMPTOTE F-JE

$\lim\limits_{x\to 0-} f(x)=-\infty \qquad \lim\limits_{x\to 0+} f(x)=+\infty$

$\Rightarrow \quad x=0$ je V.A.

$\lim\limits_{x\to -\infty} f(x)=0 \Rightarrow Y=0$ je H.A. (kad $x\to-\infty$)

$\lim\limits_{x\to +\infty} f(x)=1 \Rightarrow Y=1$ je H.A. (kad $x\to+\infty$)

F-ja nema kose asimptote

Poslije ovog koraka počinjemo skicirati graf f-je

EKSTREMI F-JE

F-ja nema ekstrema

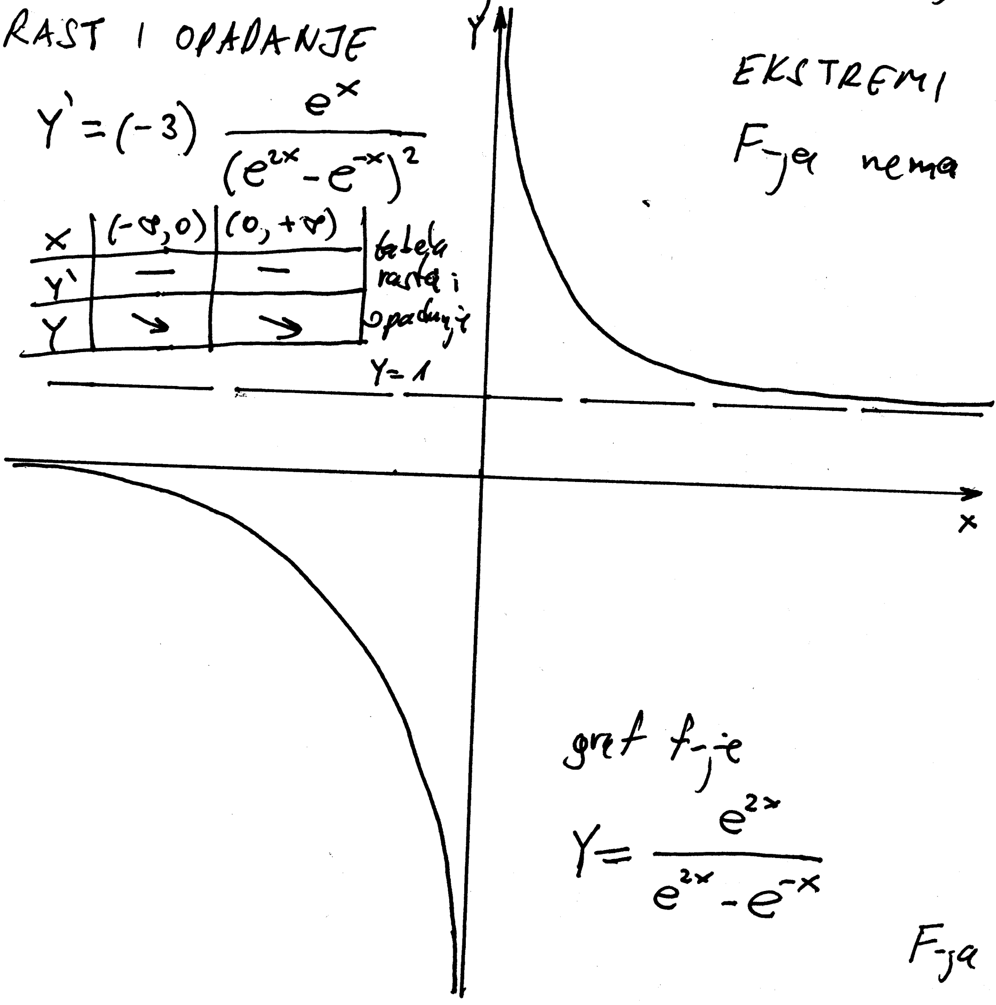
PREVOJNE TAČKE I INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$$Y''=9\frac{e^{3x}+1}{(e^{2x}-e^{-x})^3}$$

| X | $(-\infty,0)$ | $(0,+\infty)$ |
|---|---|---|
| Y'' | − | + |
| Y | ∩ | ∪ |

intervali konvek. i konkav.

F-ja nema prevojnih tački



graf f-je

$$Y=\frac{e^{2x}}{e^{2x}-e^{-x}}$$

## (#) Ispitati f-ju $Y=(2x+1)e^{-\frac{2}{x}}$ i nacrtati njen grafik.

Rj.-upute:

DEFINICIONO PODRUČJE

$\mathcal{D}: x\in(-\infty,0)\cup(0,+\infty)$

$x\in\mathbb{R}\setminus\{0\}$

NULE, PRESJEK SA Y-osom, ZNAK

$(-\frac{1}{2};0)$ je nula f-je

f-ja ne siječe Y-osu

| X | $(-\infty,-\frac{1}{2})$ | $(-\frac{1}{2},$ [illegible]$)$ | $(0,+\infty)$ |
|---|---|---|---|
| Y | − | + | + |

znak f-je

PARNOST (NEPARNOST), PERIODIČNOST

F-ja nije ni parna ni neparna.

F-ja nije periodična.

PONAŠANJE NA KRAJEVIMA INTERVALA DEFINISANOSTI I ASIMPTOTE

$\lim_{x\to 0-} f(x)=+\infty \Rightarrow x=0$ je $V_oA_o$ (sa lijeve strane)

$\lim_{x\to 0+} f(x)=0$

$\lim_{x\to\mp\infty} f(x)=\mp\infty \Rightarrow$ f-ja nema horizontalnu asimptotu

$Y=2x-3$ je $K_oA_o$

Poslije ovog koraka počinjemo sa skiciranjem grafa f-je

RAST I OPADANJE

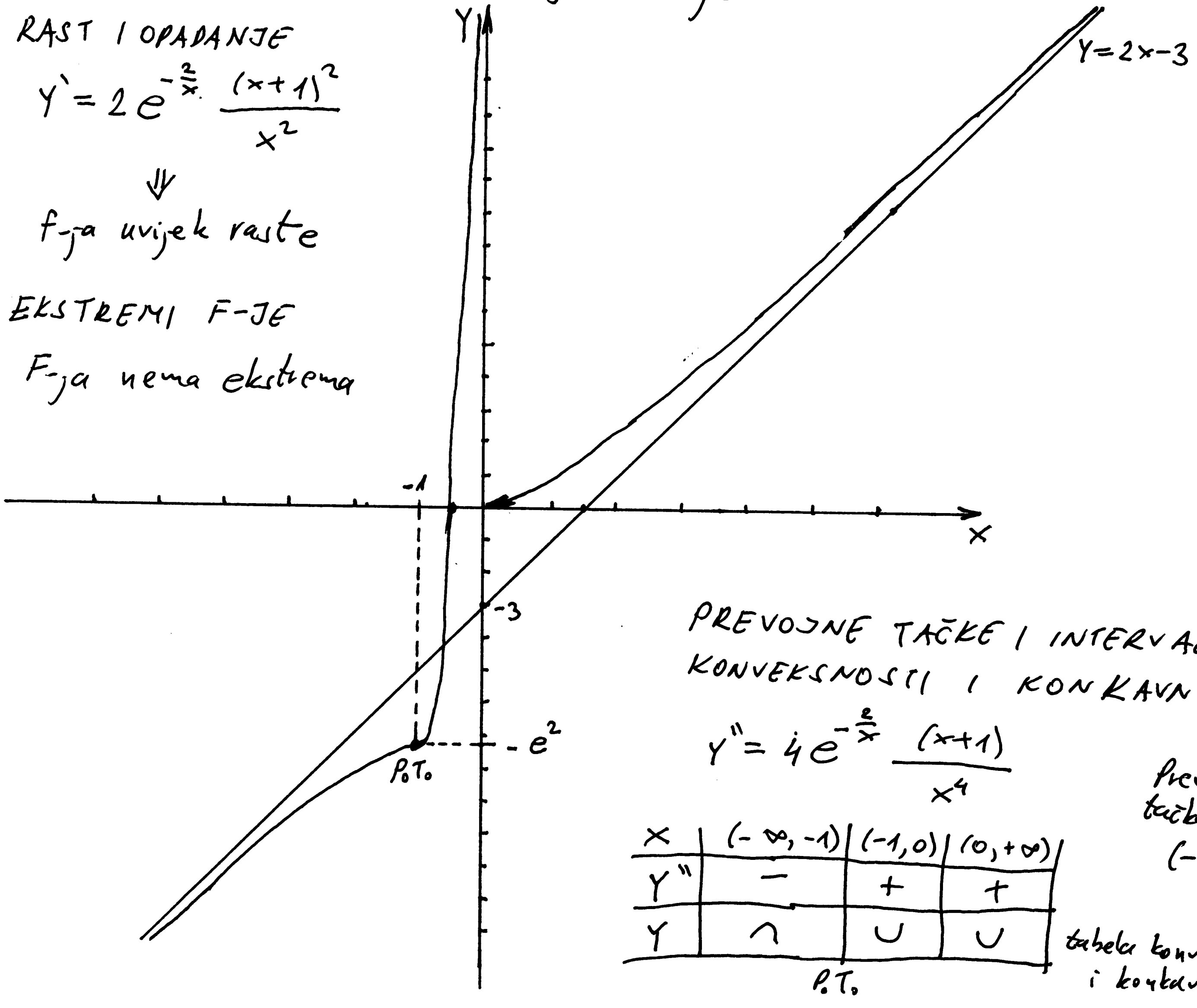
$Y'=2e^{-\frac{2}{x}}\frac{(x+1)^2}{x^2}$

$\Downarrow$

f-ja uvijek raste

EKSTREMI F-JE

F-ja nema ekstrema



PREVOJNE TAČKE I INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$Y''=4e^{-\frac{2}{x}}\frac{(x+1)}{x^4}$

Prevojna tačka je $(-1;-e^2)$

| X | $(-\infty,-1)$ | $(-1,0)$ | $(0,+\infty)$ |
|---|---|---|---|
| Y'' | − | + | + |
| Y | ∩ | ∪ | ∪ |
| | P.T. | | |

tabela konveksnosti i konkavnosti

(#) Ispitati f-ju i nacrtati njen grafik $y = \left(\frac{1}{2}x - 1\right)e^{-\frac{1}{x}}$.

Rj.-upute:

DEFINICIONO PODRUČJE

$D: x \in (-\infty, 0) \cup (0, +\infty)$

$x \in R \setminus \{0\}$

PARNOST (NEPARNOST), PERIODIČNOST

F-ja nije ni parna ni neparna.

F-ja nije periodična

NULE, PRESJEK SA Y-osom, ZNAK

(2;0) je nula f-je

f-ja ne siječe Y-osu

| X | $(-\infty, 0)$ | $(0, 2)$ | $(2, +\infty)$ |
|---|---|---|---|
| Y | − | − | + |

znak f-je

PONAŠANJE NA KRAJEVIMA INTERVALA DEFINISANOSTI I ASIMPTOTE

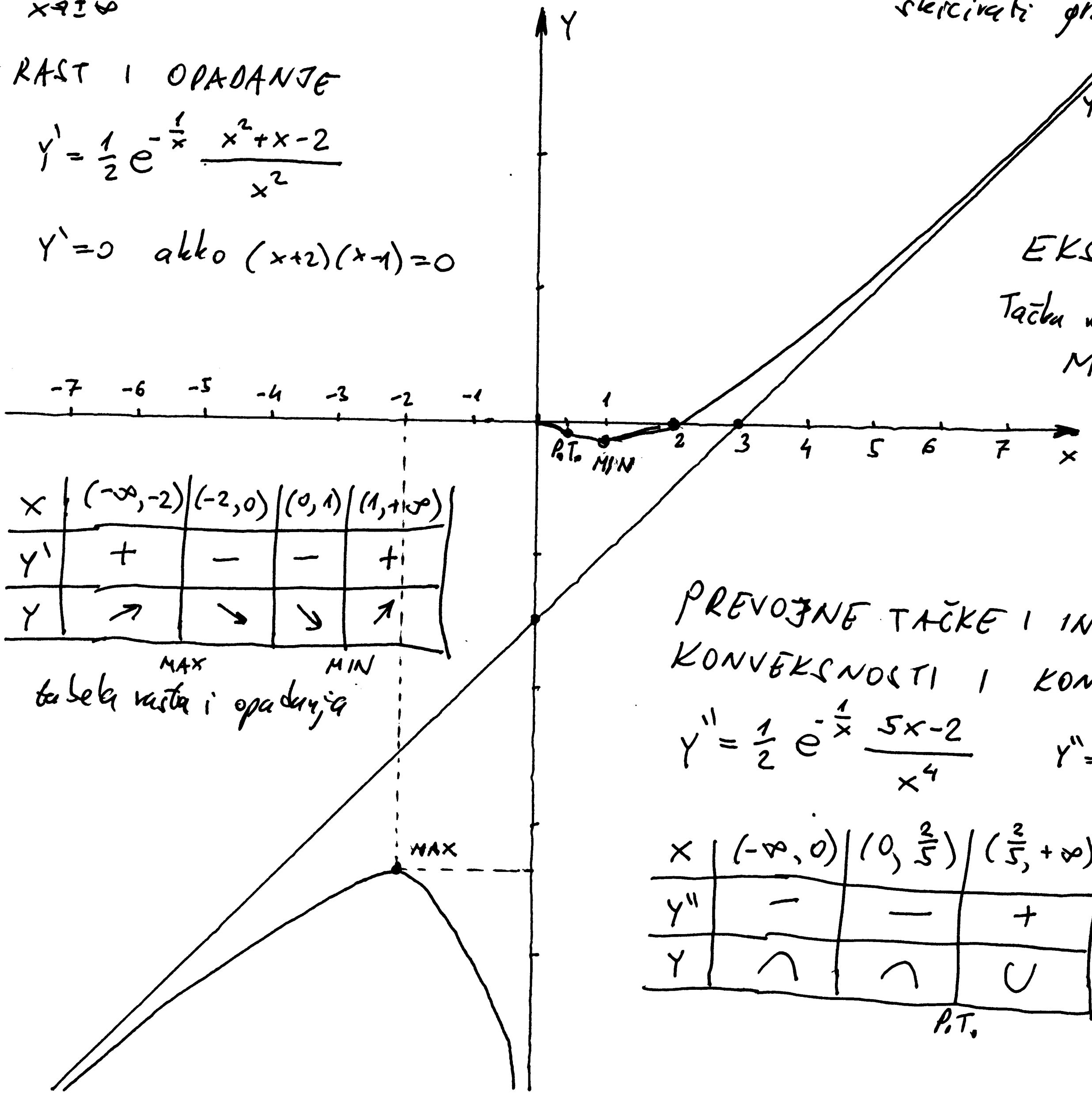
$\lim\limits_{x \to 0-} f(x) = -\infty \Rightarrow x = 0$ je V.A. (sa lijeve strane

$\lim\limits_{x \to 0+} f(x) = 0$

$\lim\limits_{x \to \pm\infty} f(x) = \pm\infty \Rightarrow$ f-ja nema H.A.

$y = \frac{1}{2}x - \frac{3}{2}$ je K.A.

Poslije ovog koraka počinjemo skicirati graf f-je.



RAST I OPADANJE

$y' = \frac{1}{2}e^{-\frac{1}{x}}\frac{x^2 + x - 2}{x^2}$

$y' = 0$ akko $(x+2)(x-1) = 0$

| X | $(-\infty, -2)$ | $(-2, 0)$ | $(0, 1)$ | $(1, +\infty)$ |
|---|---|---|---|---|
| Y' | + | − | − | + |
| Y | ↗ | ↘ | ↘ | ↗ |
| | | MAX | | MIN |

tabela rasta i opadanja

EKSTREMI F-JE

Tačka maksimuma je $M(-2; -2\sqrt{e})$

Tačka minimuma je $N(1; -\frac{1}{2}e^{-1})$

PREVOJNE TAČKE I INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$y'' = \frac{1}{2}e^{-\frac{1}{x}}\frac{5x - 2}{x^4}$ $\quad y'' = 0$ akko $x = \frac{2}{5}$

| X | $(-\infty, 0)$ | $(0, \frac{2}{5})$ | $(\frac{2}{5}, +\infty)$ |
|---|---|---|---|
| Y'' | − | − | + |
| Y | ∩ | ∩ | ∪ |
| | | | P.T. |

tabela konveksnosti i konkavnosti

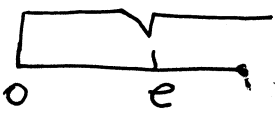
(#) Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $y=\frac{\ln x}{x}$.

Rj. -upute:

D.p. $x>0$

$x\in(0,+\infty)$

$y'=0$ akko $x=e$



prekidi y + nule y'

$$y'=\frac{1-\ln x}{x^2}$$

| x | $(0,e)$ | $(e,+\infty)$ |
|---|---|---|
| y' | + | − |
| y | ↗ | ↘ |

MAX

tabela rasta i opadanja

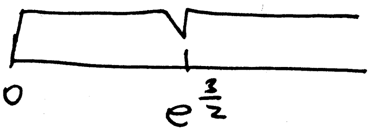
$$MAX\left(e,\frac{1}{e}\right)$$

$$y''=\frac{-3+2\ln x}{x^3}$$

$y''=0$ akko $-3+2\ln x=0$

$2\ln x=3$

$x=e^{\frac{3}{2}}$



prekidi y + nule y''

| x | $(0,e^{\frac{3}{2}})$ | $(e^{\frac{3}{2}},+\infty)$ |
|---|---|---|
| y'' | − | + |
| y | ∩ | ∪ |

P.T.

tabela konveksnosti i konkavnosti

$$P.T.\left(e^{\frac{3}{2}},\frac{3}{2e^{\frac{3}{2}}}\right)$$

(#) Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $y=\frac{1+\ln x}{x^2}$.

Rj.-uputa:

D.p. $x>0$

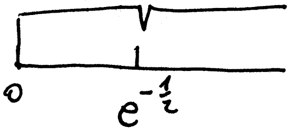
$x\in(0,+\infty)$

$y'=0$ akko $2\ln x=-1$

$\ln x=-\frac{1}{2}$

$x=e^{-\frac{1}{2}}$

$$y'=\frac{-1-2\ln x}{x^3}$$



← prekidi y + nule y'

$$f(e^{-\frac{1}{2}})=\frac{1-\frac{1}{2}}{(e^{-\frac{1}{2}})^2}$$

| x | $(0,e^{-\frac{1}{2}})$ | $(e^{-\frac{1}{2}},+\infty)$ |
|---|---|---|
| y' | + | − |
| y | ↗ | ↘ |

tabela rasta i opadanja

MAX

$e^{-\frac{7}{2}}\in(0,e^{-\frac{1}{2}})$

$e\in(e^{-\frac{1}{2}},+\infty)$

$e^{-\frac{7}{2}}<e^{-\frac{1}{2}}<e$

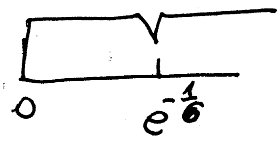
$MAX\left(e^{-\frac{1}{2}},\frac{1}{2}e\right)$

$$y''=\frac{1+6\ln x}{x^4}$$

$y''=0$ akko $6\ln x=-1$

$\ln x=-\frac{1}{6}$

$x=e^{-\frac{1}{6}}$



← prekidi y + nule y''

$e^{-1}<e^{-\frac{1}{6}}<e$

| x | $(0,e^{-\frac{1}{6}})$ | $(e^{-\frac{1}{6}},+\infty)$ |
|---|---|---|
| y'' | − | + |
| y | ∩ | ∪ |

intervali konveksnosti i konkavnosti

P.T.

$$f(e^{-\frac{1}{6}})=\frac{1-\frac{1}{6}}{e^{-\frac{1}{3}}}$$

$$P.T.\left(e^{-\frac{1}{6}},\frac{5}{6}\sqrt[3]{e}\right)$$

(#) Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $y=\frac{2+\ln x}{6x^2}$.

Rj. - upute

D.p. $x>0$

$x\in(0,+\infty)$

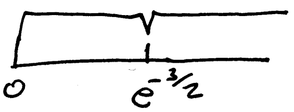
$y'=0$ akko $2\ln x+3=0$

$2\ln x=-3$

$\ln x=-\frac{3}{2}$

$x=e^{-\frac{3}{2}}$

$$y'=-\frac{2\ln x+3}{6x^3}$$



prekidi y + nule y'

| $x$ | $(0,e^{-\frac{3}{2}})$ | $(e^{-\frac{3}{2}},+\infty)$ |
|---|---|---|
| $y'$ | $+$ | $-$ |
| $y$ | ↗ | ↘ |
| | MAX | |

tabela rasta i opadanja

$$e^{-\frac{4}{2}}<e^{-\frac{3}{2}}<e$$

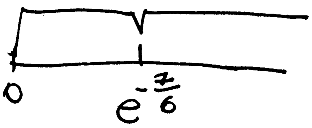
$$f(e^{-\frac{3}{2}})=\frac{2-\frac{3}{2}}{6\cdot(e^{-\frac{3}{2}})^2}=\frac{-\frac{1}{2}}{6e^{-3}}$$

$$MAX\left(e^{-\frac{3}{2}};-\frac{1}{12e^{-3}}\right)=\left(e^{-\frac{3}{2}};-\frac{e^3}{12}\right)$$

$$y''=\frac{\ln x+\frac{7}{6}}{x^4}$$

$y''=0$ akko $\ln x+\frac{7}{6}=0$

$\ln x=-\frac{7}{6}\Rightarrow x=e^{-\frac{7}{6}}$



$$e^{-\frac{9}{6}}<e^{-\frac{7}{6}}<e$$

| $x$ | $(0,e^{-\frac{7}{8}})$ | $(e^{-\frac{7}{6}},+\infty)$ |
|---|---|---|
| $y''$ | $-$ | $+$ |
| $y$ | $\cap$ | $\cup$ |
| | P.T. | |

tabela konveksnosti i konkavnosti

$$f(e^{-\frac{7}{6}})=\frac{2-\frac{7}{6}}{6(e^{-\frac{7}{6}})^2}=\frac{\frac{5}{6}}{6e^{-\frac{7}{3}}}=\frac{5}{36}e^{\frac{7}{3}}$$

$$P.T.\left(e^{-\frac{7}{6}};\frac{5}{36}e^{\frac{7}{3}}\right)$$

(#) Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $Y = \frac{1+\ln x}{\ln x}$

Rj.: - upute:

D.p. $x>0,\ \ln x \neq 0$

$x \in (0,1) \cup (1,+\infty)$

$$Y' = -\frac{1}{x \ln^2 x}$$

Y' nema nule što znači da f-ja nema ekstrem

| x | $(0,1)$ | $(1,+\infty)$ |
|---|---|---|
| Y' | − | − |
| Y | ↘ | ↘ |

tabela rasta i opadanja

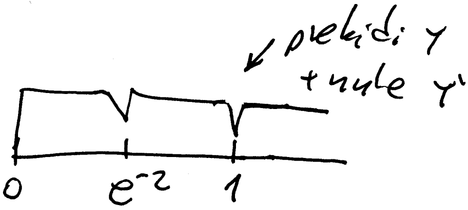
f-ja uvijek opada

$$Y'' = \frac{\ln x + 2}{x^2 \ln^3 x}$$

$Y'' = 0$ akko $\ln x + 2 = 0$

$\ln x = -2$

$x = e^{-2}$



| x | $(0,e^{-2})$ | $(e^{-2},\ \overset{e^0}{\overset{\shortparallel}{1}})$ | $(1,+\infty)$ |
|---|---|---|---|
| Y'' | + | − | + |
| Y | ∪ | ∩ | ∪ |

P.T.

tabela konveksnosti i konkavnosti

$$e^{-3} < e^{-2} < \underbrace{e^{-\frac{1}{2}}}_{\in(e^{-2},\,e^0)} < 1 < e$$

$$f(e^{-2}) = \frac{1-2}{-2} = \frac{1}{2}$$

$$P_0T_0\left(e^{-2}, \frac{1}{2}\right)$$

(#) Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $y=\frac{3+\ln x}{x}$.

Rj.-upute

Dp. $x>0$, $x\neq 0$

$x\in(0,\infty)$

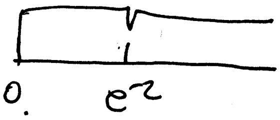
$y'=-\frac{\ln x+2}{x^2}$

$y'=0$ akko $\ln x+2=0$

$\ln x=-2$

$x=e^{-2}$

prekidi y + nule y'



| x | $(0,e^{-2})$ | $(e^{-2},+\infty)$ |
|---|---|---|
| y' | + | — |
| y | ↗ | ↘ |

MAX

tabela rasta i opadanja

$f(e^{-2})=\frac{3-2}{e^{-2}}=e^2$

$MAX(e^{-2};e^{2})$

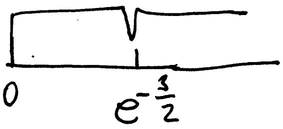
$y''=\frac{2\ln x+3}{x^3}$

$y''=0$ akko $2\ln x+3=0$

$2\ln x=-3$

$\ln x=-\frac{3}{2} \Rightarrow x=e^{-\frac{3}{2}}$

prekidi y + nule y'



| x | $(0,e^{-\frac{3}{2}})$ | $(e^{-\frac{3}{2}},+\infty)$ |
|---|---|---|
| y'' | — | + |
| y | ∩ | ∪ |

$P_oT_o$

tabela konveksnosti i konkavnosti

$f(e^{-\frac{3}{2}})=\frac{3-\frac{3}{2}}{e^{-\frac{3}{2}}}$

$P_oT_o(e^{-\frac{3}{2}};\frac{3}{2}e^{\frac{3}{2}})$

# ⑧ Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti f-je $y=\frac{1-\ln x}{x^2}$.

Rj. -upute

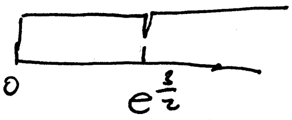
D.p. $x>0$

$x\in(0,+\infty)$

$y'=0$ akko $2\ln x=3$

$\ln x=\frac{3}{2} \Rightarrow x=e^{\frac{3}{2}}$

$$y'=\frac{2\ln x-3}{x^3}$$



← prekidi y + nule y'

| x | $(0,e^{\frac{3}{2}})$ | $(e^{\frac{3}{2}},+\infty)$ |
|---|---|---|
| y' | − | + |
| y | ↘ | ↗ |

MAX

tabela rasta i opadanja

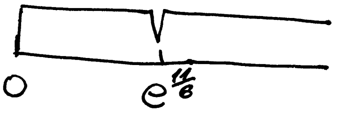
$$f(e^{\frac{3}{2}})=\frac{1-\frac{3}{2}}{(e^{\frac{3}{2}})^2}=\frac{-\frac{1}{2}}{e^3}$$

$$MAX\left(e^{\frac{3}{2}};-\frac{1}{2e^3}\right)$$

$$y''=-\frac{6\ln x-11}{x^4}$$

$y''=0$ akko $6\ln x-11=0$

$\ln x=\frac{11}{6} \Rightarrow x=e^{\frac{11}{6}}$



| x | $(0,e^{\frac{11}{6}})$ | $(e^{\frac{11}{6}},+\infty)$ |
|---|---|---|
| y'' | + | − |
| y | ∪ | ∩ |

P.T.

tabela konveksnosti i konkavnosti

$$f(e^{\frac{11}{6}})=\frac{1-\frac{11}{6}}{(e^{\frac{11}{6}})^2}=\frac{-\frac{5}{6}}{e^{\frac{11}{3}}}$$

$$P.T.\left(e^{\frac{11}{6}};-\frac{5}{6e^{11/3}}\right)$$

(#) Ispitati f-ju i nacrtati njen grafik: $y=\frac{\ln^2 x+1}{x^2}$.

R.: definiciono područje

$x\neq 0$ ; $x>0$

$\mathcal{D}: x\in(0,+\infty)$

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično

$\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$ akko $\ln^2 x+1=0$

$(\ln x)^2=-1$

f-ja nema nulu

$f(0)$ nije definisano

f-ja ne siječe y-osu

$\ln^2 x+1>0 \ \forall x\in\mathcal{D}$

$x^2>0 \ \forall x\in\mathcal{D}$

f-ja je uvijek pozitivna

ponašanje na krajevima intervala definisanosti i asimptote

za $x\leq 0$ f-ja nije definisana

$$\lim_{x\to 0+} f(x)=\lim_{x\to 0+}\frac{\ln^2 x+1}{x^2}=\frac{+\infty}{0+}=+\infty \Rightarrow x=0 \text{ je vertikalna asimptota}$$

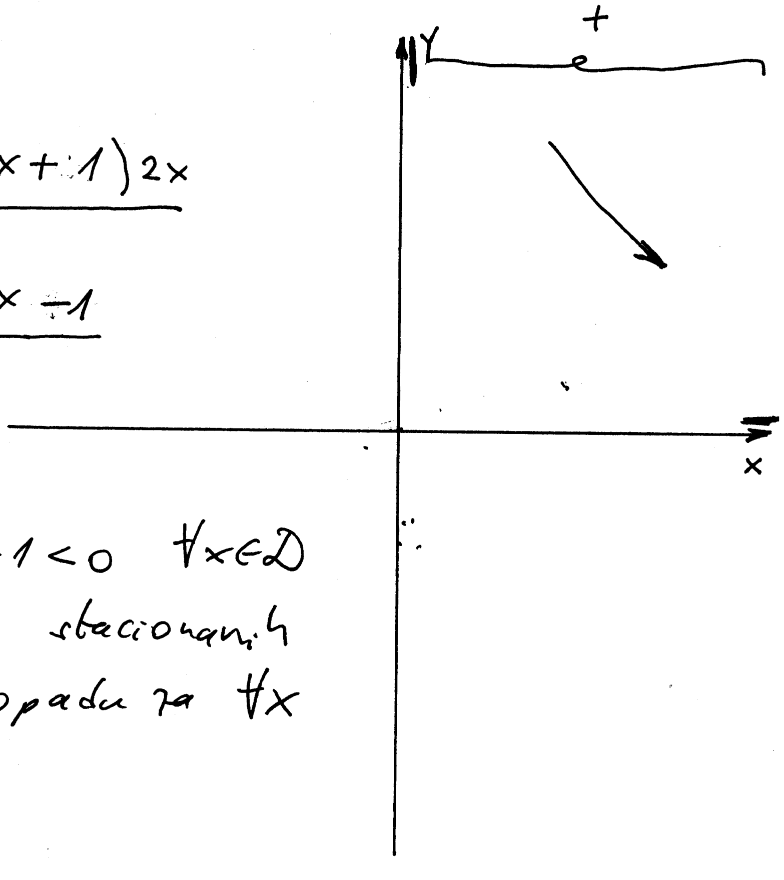
$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{\ln^2 x+1}{x^2}\left(=\frac{\infty}{\infty}\right)\overset{L.P.}{=}\lim_{x\to+\infty}\frac{2\ln x\cdot\frac{1}{x}}{2x}=\lim_{x\to+\infty}\frac{\ln x}{x^2}\left(\frac{\infty}{\infty}\right)\overset{L.P.}{=}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{x}}{2x}=0 \Rightarrow y=0 \text{ je horizontalna asimptota}$$

f-ja nema kosu asimptotu

počinjemo skicirati grafik

rast i opadanje



$$y'=\left(\frac{\ln^2 x+1}{x^2}\right)'=\frac{2\ln x\cdot\frac{1}{x}\cdot x^2-(\ln^2 x+1)2x}{x^4}$$

$$=\frac{2x(\ln x-\ln^2 x-1)}{x^4}=2\,\frac{\ln x-\ln^2 x-1}{x^3}$$

$y'=0$ akko $-\ln^2 x+\ln x-1=0$

$\ln x=t$

$-t^2+t-1=0$

$t^2-t+1=0$

$D=1-4<0$

$-\ln^2 x+\ln x-1<0 \ \forall x\in\mathcal{D}$

f-ja nema stacionarnih tački i opada za $\forall x$

ekstremi f-je

f-ja nema stacionarnih tački ⇒ f-ja nema ekstrema

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = 2\left(\frac{\ln x - \ln^2 x - 1}{x^3}\right)' = 2\,\frac{\left(\frac{1}{x} - 2\ln x\cdot\frac{1}{x}\right)x^{\cancel{3}} - (\ln x - \ln^2 x - 1)\cdot 3\cancel{x^2}}{x^{\cancel{6}4}} =$$

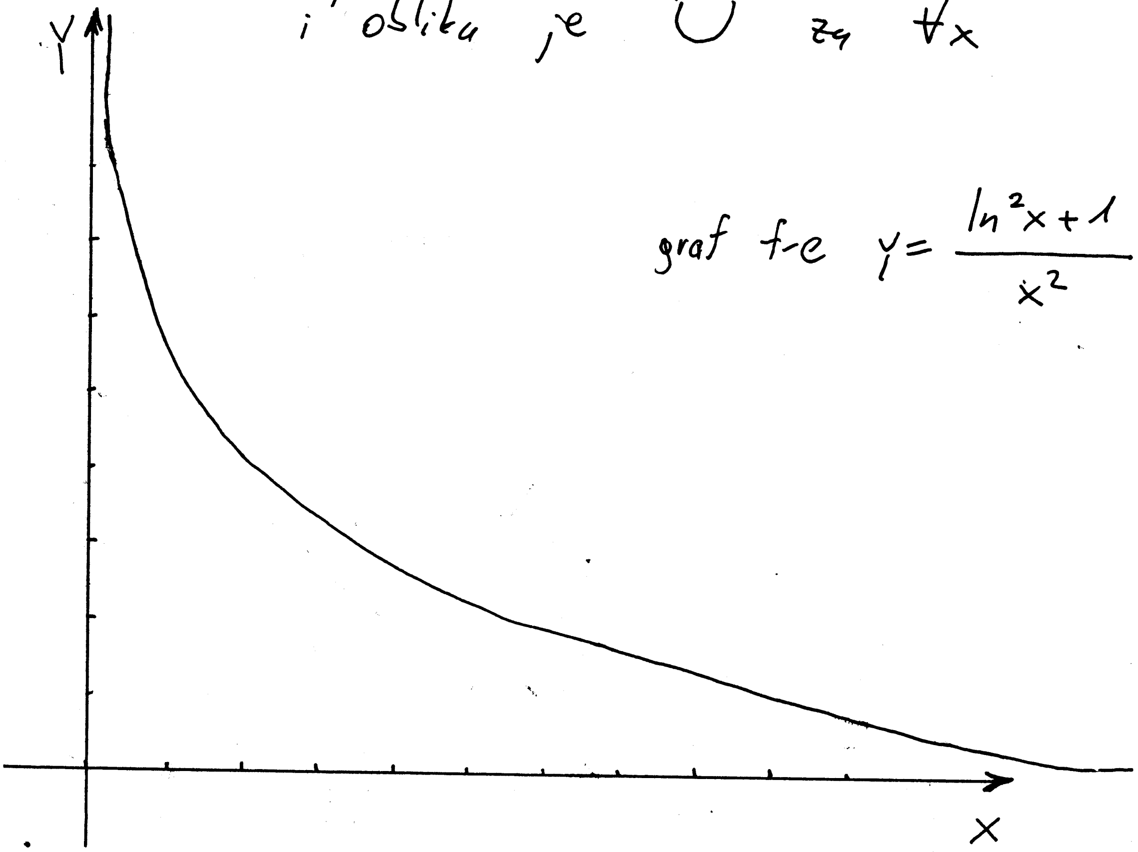
$$= 2\,\frac{1 - 2\ln x - 3\ln x + \underline{3\ln^2 x} + 3}{x^4} = 2\,\frac{3\ln^2 x - 5\ln x + 4}{x^4}.$$

$3\ln^2 x - 5\ln x + 4 = 0$

$\ln x = t \qquad 3t^2 - 5t + 4 = 0 \Rightarrow 3\ln^2 x - 5\ln x + 4 > 0 \ \forall x$

$D = 25 - 48 < 0 \qquad x^4 > 0 \ \forall x$

$y'' > 0 \ \forall x \in D$ ⇒ f-ja nema prevojnih tački i oblika je $\cup$ za $\forall x$



graf f-je $y = \dfrac{\ln^2 x + 1}{x^2}$

(#) Ispitati f-ju i nacrtati joj grafik $y = \ln \frac{x^2-3x+2}{x^2+1}$.

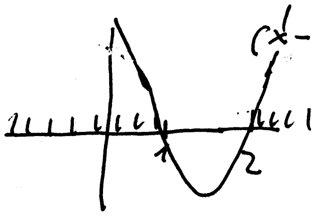
Rj. definiciono područje

Kako je $x^2+1>0 \quad \forall x \in \mathbb{R}$

to iz $\frac{x^2-3x+2}{x^2+1} > 0 \Rightarrow$

treba da bude $x^2-3x+2>0$

$(x-1)(x-2)>0$



$\mathcal{D}: x \in (-\infty, 1) \cup (2, +\infty)$

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak

$y = 0 \Rightarrow \ln \frac{x^2-3x+2}{x^2+1} = 0$

$\Rightarrow \frac{x^2-3x+2}{x^2+1} = 1 \quad / \cdot x^2+1$

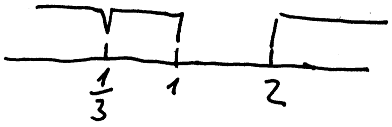
$\cancel{x^2}-3x+2 = \cancel{x^2}+1$

$3x = 1 \Rightarrow x = \frac{1}{3}$

$(\frac{1}{3}, 0)$ je nula f-je

$y(0) = \ln 2 \approx 0,6931$

$(0, \ln 2)$ je presjek sa y-osom



← prekidi y + nule y

| x | $(-\infty, \frac{1}{3})$ | $(\frac{1}{3}, 1)$ | $(2, +\infty)$ |
|---|---|---|---|
| y | + ∘ | − | − |

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

f-ja ima prekid za $x=1$ i $x=2$

$\lim_{x\to 1-0} f(x) = \lim_{x \to 1-0} \ln \frac{x^2-3x+2}{x^2+1} = \ln \frac{(1-0)^2-3(1-0)+2}{(1-0)^2+1} = \ln(0_+) = -\infty \Rightarrow$

$\Rightarrow x=1$ je V. A. (sa lijeve str.)

$\lim_{x\to 2+0} f(x) = \lim_{x \to 2+0} \ln \frac{x^2-3x+2}{x^2+1} = \ln(0+) = -\infty$

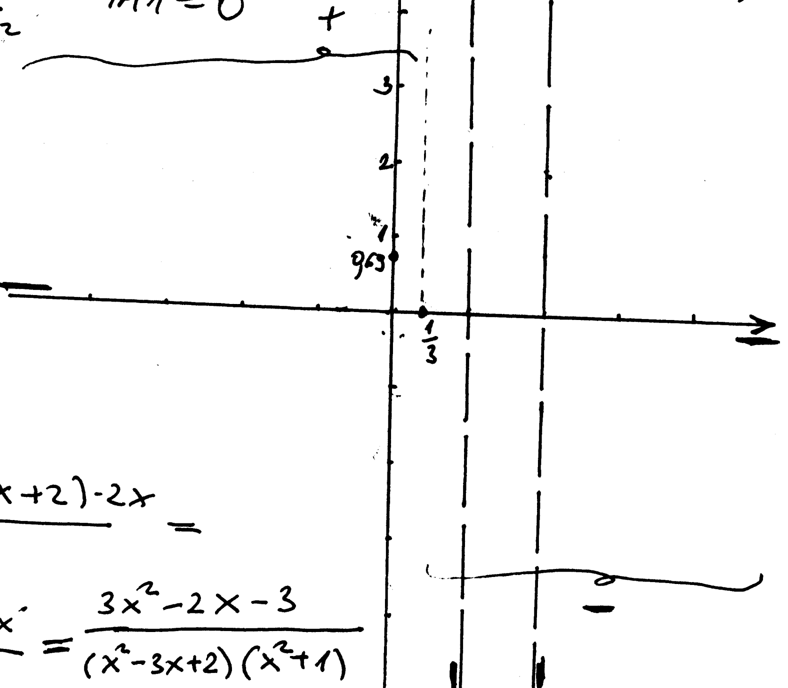
$\Rightarrow x=2$ je V. A. (sa desne strane)

$\lim_{x\to \mp\infty} f(x) = \lim_{x \to \mp\infty} \ln \frac{x^2-3x+2}{x^2+1} = \lim_{x\to\mp\infty} \frac{1-\frac{3}{x}+\frac{2}{x^2}}{1+\frac{1}{x^2}} = \ln 1 = 0$

$\Rightarrow y=0$ je H. A.

K. A. nema

počinjemo sa skiciranjem grafa



rast i opadanje

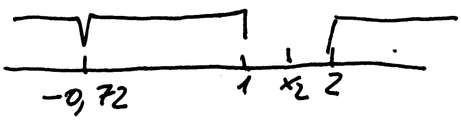
$y' = \frac{x^2+1}{x^2-3x+2} \cdot \left( \frac{x^2-3x+2}{x^2+1} \right)'$

$y' = \frac{\cancel{x^2+1}}{x^2-3x+2} \cdot \frac{(2x-3)(x^2+1)-(x^2-3x+2)\cdot 2x}{(x^2+1)^{\cancel{2}}} =$

$= \frac{\cancel{2x^3}+2x-3x^2-3-\cancel{2x^3}+6x^2-4x}{(x^2-3x+2)(x^2+1)} = \frac{3x^2-2x-3}{(x^2-3x+2)(x^2+1)}$

$Y'=0 \Rightarrow 3x^2-2x-3=0 \Rightarrow x_{1,2}=\frac{2\pm\sqrt{4+36}}{6}$

$x_{1,2}=\frac{2\pm 2\sqrt{10}}{6}=\frac{1\pm\sqrt{10}}{3}$



$x_1=\frac{1+\sqrt{10}}{3}\approx 1,387 \notin D$

$x_2=\frac{1-\sqrt{10}}{3}\approx -0,721 \in D$

| x | $(-\infty, \frac{1-\sqrt{10}}{3})$ | $(\frac{1-\sqrt{10}}{3}, 1)$ | $(2, +\infty)$ |
|---|---|---|---|
| Y' | + | − | + |
| Y | ↗ | ↘ | ↗ |

max

ekstremi f-je

$f(\frac{1-\sqrt{10}}{3})\approx 1,016$

F-ja ima maksimum u tački (-0,72; 1,02)

prevojne tačke i intervali konveksnosti i konkavnosti

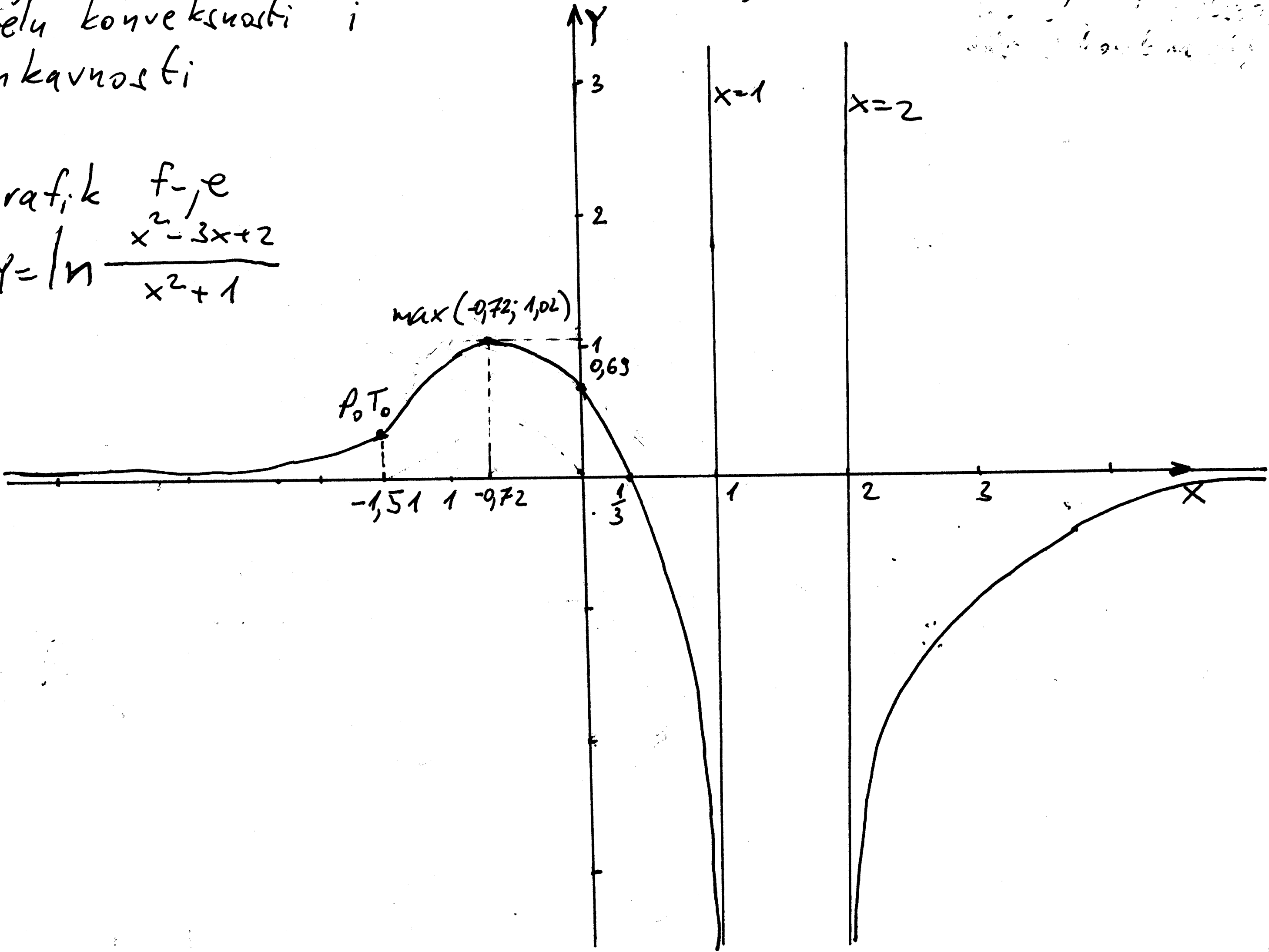
$Y''=\left(\frac{3x^2-2x-3}{(x^2-3x+2)(x^2+1)}\right)' = \overset{\text{ZA}}{\text{VJEŽBU}} \ldots = \frac{-6x^5+15x^4-30x^2+30x-13}{(x^2-3x+2)^2(x^2+1)^2}$

$Y''=0$ akko $x=-1,5166$ (izračunato uz pomoć kalkulatora)

Kako je brojnik u $Y''$ previše složen nije potrebno praviti tabelu konveksnosti i konkavnosti

grafik f-je

$Y=\ln\frac{x^2-3x+2}{x^2+1}$



#) Odrediti definiciono područje, znak te ekstreme f-je

$$y = \ln \frac{x}{x^2-1}$$

Rj.-upute

DEFINICIONO PODRUČJE

$x^2-1 \neq 0 \quad \wedge \quad \frac{x}{x^2-1} > 0$

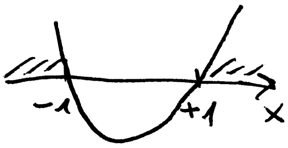
$x^2 \neq 1$

$x \neq \pm 1$

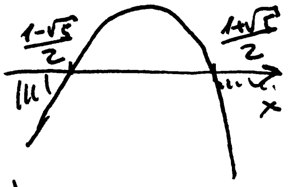
$\mathcal{D}: x \in (-1,0) \cup (1,+\infty)$

| x | $(-\infty,-1)$ | $(-1,0)$ | $(0,1)$ | $(1,+\infty)$ |
|---|---|---|---|---|
| $x$ | $-$ | $-$ | $+$ | $+$ |
| $x^2-1$ | $+$ | $-$ | $-$ | $+$ |
| $\frac{x}{x^2-1}$ | $-$ | ⊕ | $-$ | ⊕ |

$x^2-1=0$



$-x^2+x+1$



ZNAK

$\ln \frac{x}{x^2-1} > 0 \qquad \frac{x}{x^2-1} - 1 > 0 \qquad \frac{(-1)(x^2-x-1)}{x^2-1} > 0$

$\ln \frac{x}{x^2-1} > \ln 1 \qquad \frac{x-(x^2-1)}{x^2-1} > 0 \qquad x^2-x-1>0 \quad D = 1+4 = 5$

$x_{1,2} = \frac{1 \pm \sqrt{5}}{2}$

$\frac{x}{x^2-1} > 1 \qquad \frac{-x^2+x+1}{x^2-1} > 0$

| X | $\left(-1, \frac{1-\sqrt{5}}{2}\right)$ | $\left(\frac{1-\sqrt{5}}{2}, 0\right)$ | $\left(1, \frac{1+\sqrt{5}}{2}\right)$ | $\left(\frac{1+\sqrt{5}}{2}, +\infty\right)$ |
|---|---|---|---|---|
| Y | $+$ | $-$ | $+$ | $-$ |

znak f-je

EKSTREMI F-JE

$$y' = \left(\ln \frac{x}{x^2-1}\right)' = \frac{x^2+1}{-x^3+x} = \frac{x^2+1}{(-x)(x^2-1)}$$

| x | $(-1,0)$ | $(1,+\infty)$ |
|---|---|---|
| $y'$ | $-$ | $-$ |
| $y$ | ↘ | ↘ |

F-ja uvijek opada pa nema ekstrema.

(#) Odrediti definiciono područje, znak te ekstreme f-je

$$y = \ln \frac{x-1}{x^2+1}$$

Rj.-upute

DEFINICIONO PODRUČJE

$x^2+1$ je pozitivno za svako $x \in R$

pa je $\frac{x-1}{x^2+1} > 0$ akko $x-1>0$

tj. za $x>1$

$\mathcal{D}: x \in (1,+\infty)$

$x > 1$

ZNAK

$$\ln \frac{x-1}{x^2+1} > 0$$

$$\ln \frac{x-1}{x^2+1} > \ln 1$$

$$\frac{x-1}{x^2+1} > 1$$

$$\frac{x-1}{x^2+1} - 1 > 0$$

$$\frac{x-1-(x^2+1)}{x^2+1} > 0$$

$$\frac{-x^2+x-2}{x^2+1} > 0$$

$$\frac{(-1)(x^2-x+2)}{x^2+1} > 0$$

Kako je $x^2-x+2>0 \ \forall x$

to je $(-1)(x^2-x+2)<0 \ \forall x$

| X | $(1,+\infty)$ |
|---|---|
| Y | − |

Znak f-je

EKSTREMI F-JE

$$y' = \left(\ln \frac{x-1}{x^2+1}\right)' = -\frac{x^2-2x-1}{(x^2+1)(x-1)}$$

$x^2-2x-1=0$

$D = 4+4=8$

$x_{1,2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

$x_1 = 1-\sqrt{2} \notin \mathcal{D}$ $\quad x_2 = 1+\sqrt{2} \in \mathcal{D}$

| x | $(1, 1+\sqrt{2})$ | $(1+\sqrt{2}, +\infty)$ |
|---|---|---|
| y' | + | − |
| y | ↗ | ↘ |

MAX

tabela rasta i opadanja

$$f(1+\sqrt{2}) = \ln \frac{1+\sqrt{2}-1}{(1+\sqrt{2})^2+1} =$$

$$= \ln \frac{\sqrt{2}}{1+2\sqrt{2}+2+1} = \ln \frac{\sqrt{2}}{4+2\sqrt{2}}$$

F-ja ima ekstrem i to maksimum u tački $\left(1+\sqrt{2}; \ln \frac{\sqrt{2}}{4+2\sqrt{2}}\right)$

# ⑂ Odrediti definiciono područje, znak te ekstreme f-je

$$y = \ln\frac{x^2-1}{x+1}.$$

Rj.--upute

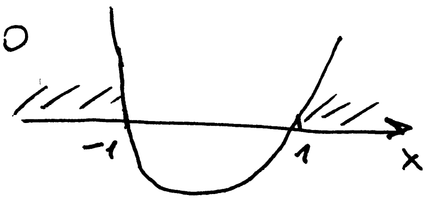
DEFINICIONO PODRUČJE

$x+1\neq 0 \quad \wedge \quad \frac{x^2-1}{x+1} > 0$

$x\neq -1$

$\frac{(x-1)(x+1)}{x+1} > 0$

$x^2-1>0$



| $x$ | $(-\infty,-1)$ | $(-1,1)$ | $(1,+\infty)$ |
|---|---|---|---|
| $x^2-1$ | + | − | + |
| $x+1$ | − | + | + |
| $\frac{x^2-1}{x+1}$ | − | − | ⊕ |

$\mathcal{D}:\ x\in(1,+\infty)$

$x>1$

ZNAK

$\ln\frac{x^2-1}{x+1} > 0$

$\ln\frac{x^2-1}{x+1} > \ln 1$

$\frac{x^2-1}{x+1} > 1$

$\frac{x^2-1}{x+1} - 1 > 0$

| $x$ | $(2,+\infty)$ |
|---|---|
| $\frac{x^2-x-2}{x+1}$ | + |

$\frac{x^2-1-x-1}{x+1} > 0$

$\frac{x^2-x-2}{x+1} > 0$

$\frac{(x-2)(x+1)}{x+1} > 0$

2, -1

| $x$ | $(1,2)$ | $(2,+\infty)$ |
|---|---|---|
| $Y$ | − | + |

Znak f-je

EKSTREMI F-JE

$$Y' = \left(\ln\frac{x^2-1}{x+1}\right)' = \frac{1}{x-1}$$

| $x$ | $(1,+\infty)$ |
|---|---|
| $Y'$ | + |
| $Y$ | ↗ |

tabela rasta i opadanja

F-ja uvijek raste pa nema ekstrema.

(#) Odrediti definiciono područje, znak te ekstreme f-je

$$y = \ln \frac{x+1}{x-1}.$$

Rj.-upute:

DEFINICIONO PODRUČJE

$x-1 \neq 0$

$x \neq 1$

$\frac{x+1}{x-1} > 0$

| $x$ | $(-\infty,-1)$ | $(-1,1)$ | $(1,+\infty)$ |
|---|---|---|---|
| $x+1$ | $-$ | $+$ | $+$ |
| $x-1$ | $-$ | $-$ | $+$ |
| $\frac{x+1}{x-1}$ | ⊕ | $-$ | ⊕ |

$\mathcal{D}: x \in (-\infty,-1) \cup (1,+\infty)$

ZNAK

$$\ln \frac{x+1}{x-1} > 0$$

$$\ln \frac{x+1}{x-1} > \ln 1$$

$$\frac{x+1}{x-1} > 1$$

$$\frac{x+1}{x-1} - 1 > 0$$

$$\frac{x+1-x+1}{x-1} > 0$$

$$\frac{2}{x-1} > 0$$

$x-1 > 0$

$x > 1$

| $x$ | $(-\infty,-1)$ | $(1,+\infty)$ |
|---|---|---|
| $y$ | $-$ | $+$ |

znak f-je

EKSTREMI F-JE

$$y' = \left(\ln \frac{x+1}{x-1}\right)' = -\frac{2}{x^2-1}$$

| $x$ | $(-\infty,-1)$ | $(1,+\infty)$ |
|---|---|---|
| $y'$ | $-$ | $-$ |
| $y$ | ↘ | ↘ |

tabela rasta i opadanja

F-ja uvijek opada pa nema ekstrema.

ⓘ Ispitati f-ju i nacrtati njen grafik $y=\frac{3x^2-1}{(x^2+1)^3}$.

Rj.-upute:

$D: x \in R$

parna f-ja (simetrična u odnosu na y-osu)

$(0,-1)$ presjek sa y-osom.

$(-\frac{1}{\sqrt{3}},0)$ i $(\frac{1}{\sqrt{3}},0)$ nule f-je

| x | $(0,\frac{1}{\sqrt{3}})$ | $(\frac{1}{\sqrt{3}},+\infty)$ |
|---|---|---|
| y | − | + |

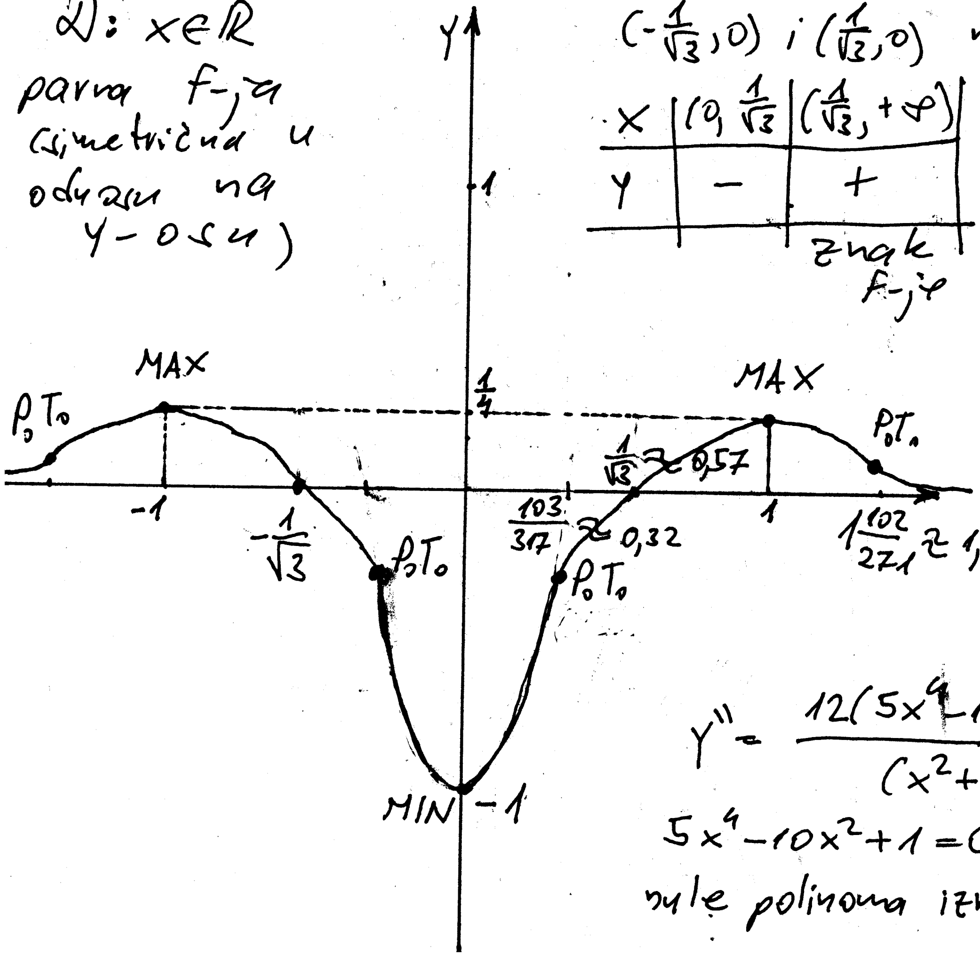
znak f-je

nema tačke prekida

⇓

nema V.A.

$y=0$ je H.A.



$$y'=\frac{-12x(x^2-1)}{(x^2+1)^4}$$

| x | $(0,1)$ | $(1,+\infty)$ |
|---|---|---|
| y' | + | − |
| y | ↗ | ↘ |

MIN $(0,-1)$ MAX $(1,\frac{1}{4})$

$$y''=\frac{12(5x^4-10x^2+1)}{(x^2+1)^5}$$

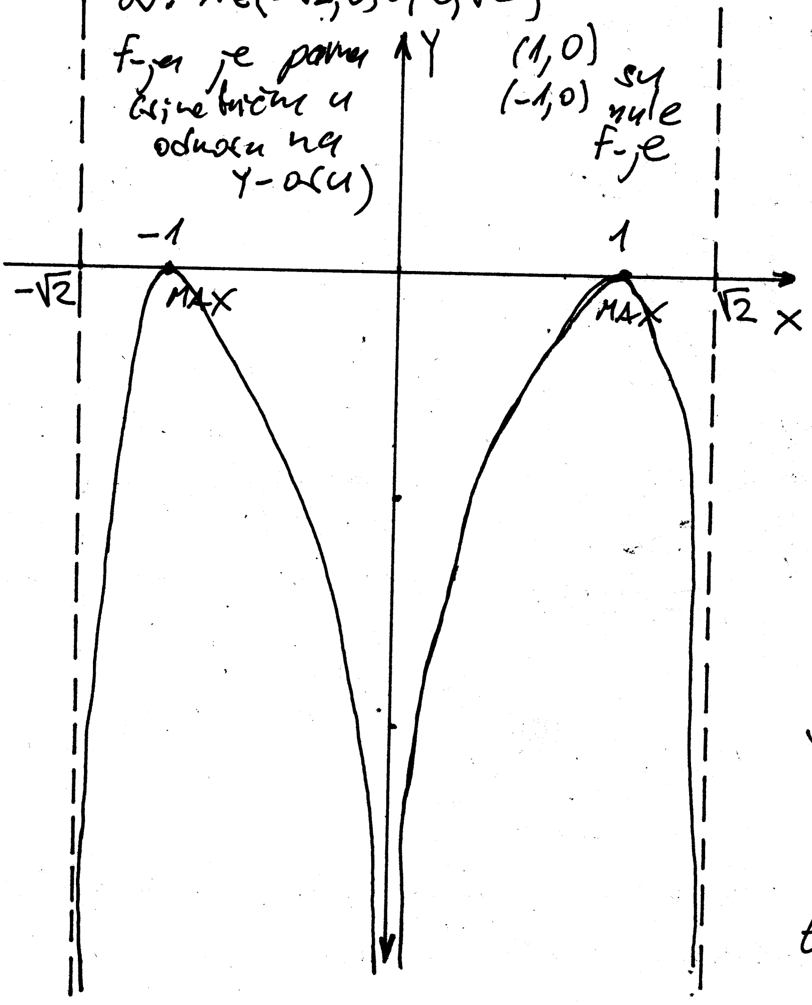
$5x^4-10x^2+1=0$ akko $x_1=\frac{103}{317}$, $x_2=1\frac{102}{271}$

nule polinoma izračunate na digitronu.

(#) Ispitati f-ju i nacrtati njen grafik $y=\ln(2x^2-x^4)$.

Rj.-upute:

$D: x\in(-\sqrt{2},0)\cup(0,\sqrt{2})$



f-ja je parna (simetrična u odnosu na y-osu)

(1,0) (-1,0) su nule f-je

$x=0$ je $V_oA_o$ nema $H_oA_o$

$x=\sqrt{2}$ je $V_oA_o$ nema $K_oA_o$

f-ja je negativna za $\forall x\in D$

$$y'=\frac{4(x^2-1)}{x(x^2-2)}$$

| x | (0,1) | (1,√2) |
|---|---|---|
| y' | + | − |
| y | ↗ | ↘ |

MAX (1,0)

$$y''=(-4)\,\frac{x^4-x^2+2}{x^2(-2+x^2)^2}$$

$y''<0\ \forall x$ f-ja nema prevojnih tački i uvijek je $\cap$

(#) Ispitati f-ju i nacrtati njen grafik $y=\frac{e^x-e^{-x}}{e^x+e^{-x}}$.

Rj.-upute:

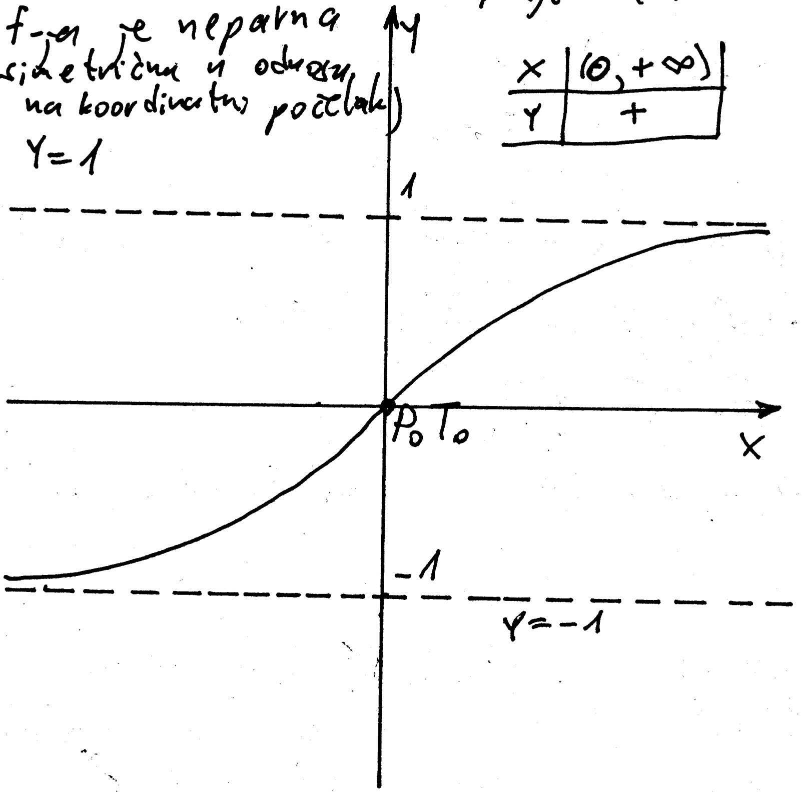
$D: x\in\mathbb{R}$

(0,0) je nula i presjek sa Y-osom

f-ja je neparna (simetrična u odnosu na koordinatni početak)

$Y=1$

| x | $(0,+\infty)$ |
|---|---|
| Y | + |



f-ja je definirana za $\forall x\in\mathbb{R}$ $\Downarrow$

nema $V_oA_o$

$Y=\pm 1$ je $H_oA_o$

$$y'=\frac{4\cdot e^{-2x}}{(1+e^{-2x})^2}=\frac{4}{(e^x+e^{-x})}$$

$y'>0$ $\forall x\in D$ f-ja raste za $\forall x\in D$

nema ekstrema

$$y''=\frac{-8(e^x-e^{-x})}{(e^x+e^{-x})^3},$$

| x | $(0,+\infty)$ |
|---|---|
| $y''$ | $-$ |
| $y$ | $\cap$ |

$P_0T_0$

$y''=0$ akko $x=0$

(0,0) je $P_oT_o$

## (#) Ispitati f-ju i nacrtati njen grafik

$$Y = \frac{3x-1}{(x^2+1)^2}.$$

R.-upute:

DEFINICIONO PODRUČJE

$D: x \in R$

$x \in (-\infty, +\infty)$

PARNOST (NEPARNOST), PERIODIČNOST

f-ja nije ni parna ni neparna

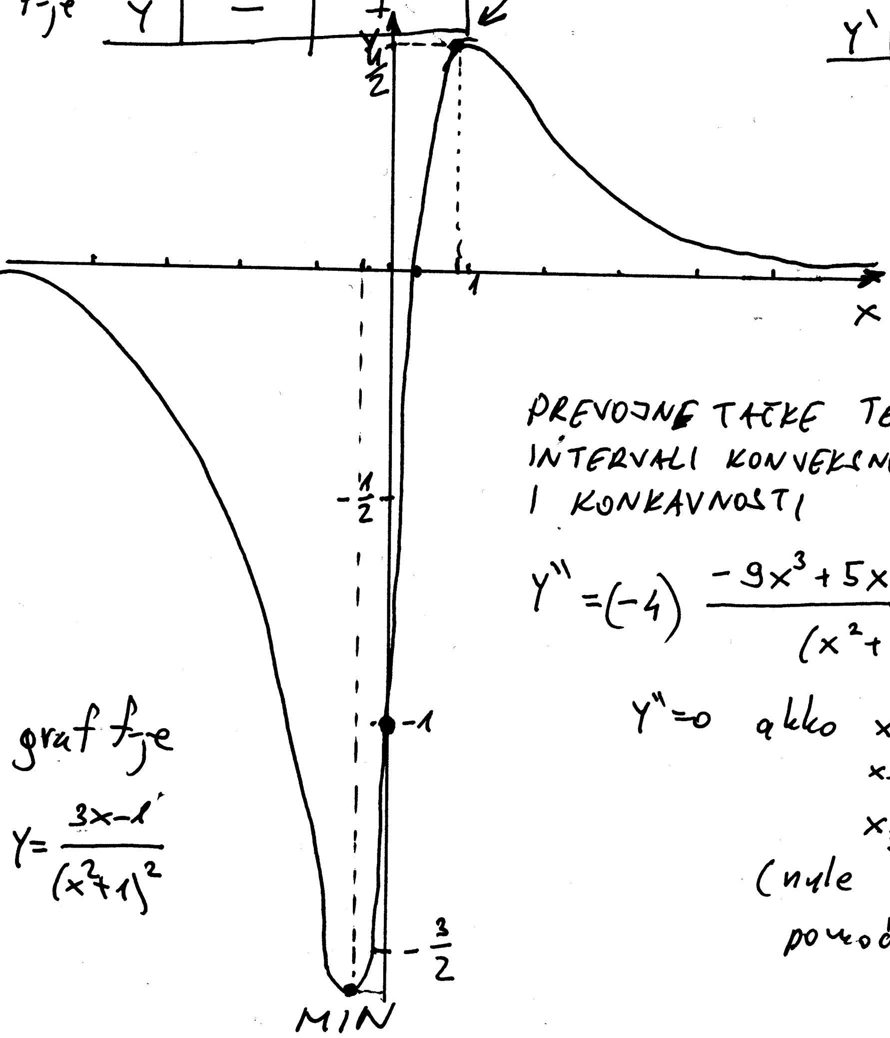
f-ja nije periodična

NULE, PRESJEK SA Y-OSOM, ZNAK

$(0,-1)$ je presjek sa y-osom

$(\frac{1}{3}, 0)$ je nula f-je

| znak f-je | X | $(-\infty, \frac{1}{3})$ | $(\frac{1}{3}, +\infty)$ |
|---|---|---|---|
| | Y | − | + |



graf f-je $Y = \frac{3x-1}{(x^2+1)^2}$

PONAŠANJE NA KRAJEVIMA INTERVALA DEFINISANOSTI I ASIMPTOTE

nema tački prekida ⇒ nema V. A.

$Y = 0$ je H. A.

Nakon ovog koraka počinjemo skicirati graf f-je

RAST I OPADANJE

$$Y' = \frac{-9x^2+4x+3}{(x^2+1)^3},$$

$D = 16 + 108 = 124$

$\sqrt{D} = 2\sqrt{31}$

$x_{1,2} = \frac{-4 \pm 2\sqrt{31}}{-18}$

0,84

-0,4

| X | $(-\infty, \frac{2-\sqrt{31}}{9})$ | $(\frac{2-\sqrt{31}}{9}, \frac{2+\sqrt{31}}{9})$ | $(\frac{2+\sqrt{31}}{9}, +\infty)$ |
|---|---|---|---|
| Y | − | + | − |
| Y' | ↘ | ↗ | ↘ |
| | MIN | MAX | |

intervali rasta i opad.

EKSTREMI F-JE

Na osnovu tabele rasta i opadanja

$$MIN\left(\frac{2-\sqrt{31}}{9}, \frac{\frac{\sqrt{31}}{3}+\frac{1}{3}}{\left(\left(\frac{2-\sqrt{31}}{9}\right)^2+1\right)^2}\right)$$

$$MAX\left(\frac{2+\sqrt{31}}{9}, \frac{\frac{\sqrt{31}-1}{3}}{\left(\left(\frac{2+\sqrt{31}}{9}\right)^2+1\right)^2}\right)$$

PREVOJNE TAČKE TE INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$$Y'' = (-4)\frac{-9x^3+5x^2+9x-1}{(x^2+1)^4}$$

$Y'' = 0$ akko $x_1 \approx 1,27$

$x_2 \approx -0,82$

$x_3 \approx 0,10$

(nule izračunate uz pomoć kalkulatora)

0,5

-1,6

## # Ispitati f-ju i nacrtati njen grafik

$$Y = \ln(2x - x^3)$$

-1,62
0,62

1+4

Rj.-upute:

DEFINICIONO PODRUČJE

$\mathcal{D}: x \in (-\infty, -\sqrt{2}) \cup (0, \sqrt{2})$

PARNOST (NEPARNOST), PERIODIČNOST

f-ja nije ni parna ni neparna

f-ja nije periodična

NULE, PRESJEK SA Y-OSOM, ZNAK

za $x=0$ f-ja nije definisana $\Rightarrow$

$\Rightarrow$ f-ja ne siječe y-osu

$Y=0 \Rightarrow \quad 2x - x^3 = 1$

$x^3 - 2x + 1 = 0$

$x=1 \Rightarrow \quad 1^3 - 2\cdot 1 + 1 = 0$

$(x^3-2x+1):(x-1) = x^2+x-1$

$-\underline{x^3 - x^2}$

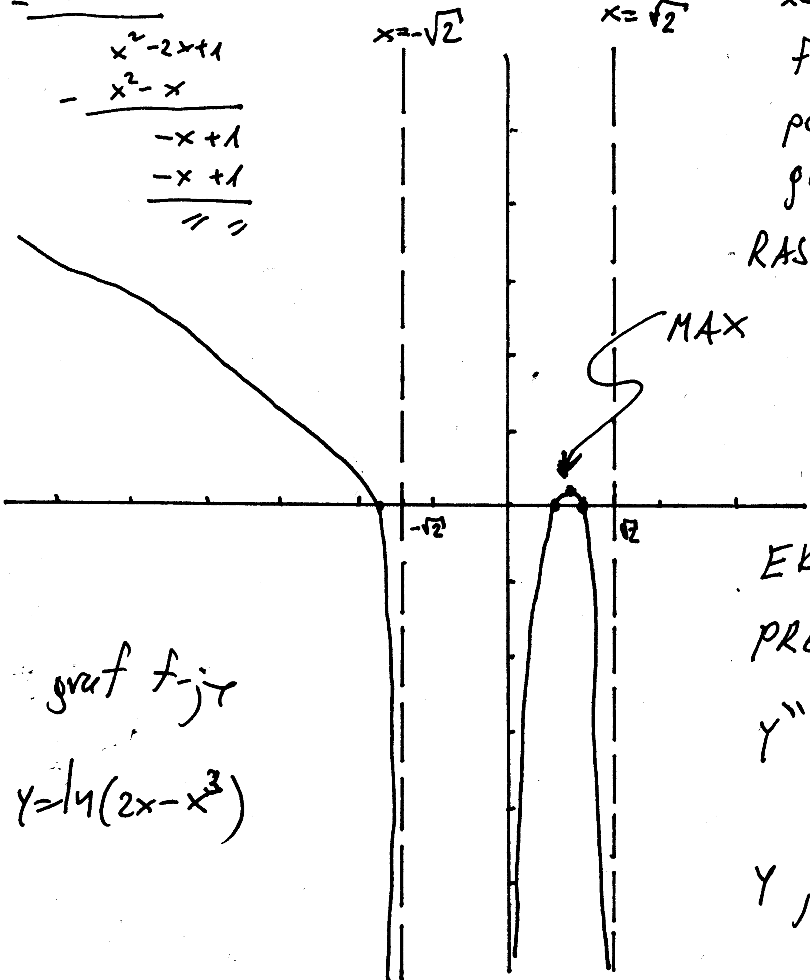
$x^2 - 2x + 1$

$-\underline{x^2 - x}$

$-x+1$

$-\underline{-x+1}$

$/\!/ \; /\!/$



graf f-je

$y = \ln(2x - x^3)$

$x^3 - 2x + 1 = (x^2+x-1)(x-1)$

$x_1 = 1, \quad x_{2,3} = \frac{-1 \pm \sqrt{5}}{2} \quad x_2 = \frac{-1-\sqrt{5}}{2}$

$N_1(1;0),\ N_2\left(\frac{-1-\sqrt{5}}{2};0\right),\ N_3\left(\frac{-1+\sqrt{5}}{2};0\right) \quad x_3 = \frac{-1+\sqrt{5}}{2}$

| x | $\left(-\infty, \frac{1}{2}(-1-\sqrt{5})\right)$ | $\left(\frac{1}{2}(-1-\sqrt{5}), -\sqrt{2}\right)$ | $\left(0, \frac{-1+\sqrt{5}}{2}\right)$ | $\left(\frac{-1+\sqrt{5}}{2}, 1\right)$ |
|---|---|---|---|---|
| Y | + | — | — | + |

| x | $(1, \sqrt{2})$ |
|---|---|
| Y | — |

znak f-je

PONAŠANJE NA KRAJEVIMA INTERVALA DEFINISANOSTI I ASIMPTOTE

$\lim\limits_{x\to-\sqrt{2}+0} f(x) = \ln(+0) = -\infty \quad \Rightarrow \quad x=-\sqrt{2}$ je V.A.

$\lim\limits_{x\to 0+} f(x) = \ln(+0) = -\infty \quad \Rightarrow \quad x=0$ je V.A.

$\lim\limits_{x\to\sqrt{2}-0} f(x) = \ln(+0) = -\infty \quad \Rightarrow \quad x=\sqrt{2}$ je V.A.

f-ja nema H.A., nema K.A.

poslije ovog koraka počinjemo skicirati graf f-je

RAST I OPADANJE $\quad Y' = -\frac{3x^2-2}{2x-x^3}$

| x | $(-\infty, -\sqrt{2})$ | $\left(0, \frac{\sqrt{2}}{\sqrt{3}}\right)$ | $\left(\frac{\sqrt{2}}{\sqrt{3}}, \sqrt{2}\right)$ |
|---|---|---|---|
| Y' | — | + | — |
| Y | ↘ | ↗ | ↘ |

tab. rasta i opad.

$\frac{\sqrt{2}}{\sqrt{3}}$

0,81

EKSTREMI $\quad Y_{max}\left(\frac{\sqrt{2}}{\sqrt{3}}\right) = \ln\frac{4\sqrt{2}}{3\sqrt{3}}$

0,084

PREV. TAČ. TE INT. KONV. I KONKAVN.

$Y'' = -\frac{3x^4+4}{x^2(x^2-2)^2} \qquad Y'' < 0 \ \forall x \in \mathcal{D}$

Y je uvijek ∩ i nema P.T.

## ⑭ Ispitati f-ju i nacrtati njen grafik

$$y=\frac{e^x}{e^x+e^{-x}}$$

Rj.-upute:

DEFINICIONO PODRUČJE

$x\in\mathbb{R}$

$x\in(-\infty,+\infty)$

PARNOST (NEPARNOST), PERIODIČNOST

f-ja nije ni parna ni neparna

f-ja nije periodična

NULE, PRESJEK SA y-OSOM, ZNAK

$y=0 \Rightarrow e^x=0$ # kontradikcija ($e^x>0\ \forall x\in\mathbb{R}$)

f-ja nema nulu

$x=0 \Rightarrow y=\frac{1}{1+1}=\frac{1}{2}$

$(0;\frac{1}{2})$ je presjek sa y-osom

Kako je $e^x>0\ \forall x$ to je

| x | $(-\infty,+\infty)$ |
|---|---|
| y | + |

znak f-je

PONAŠANJE NA KRAJEVIMA INTERVALA DEFINISANOSTI I ASIMPTOTE

f-ja je neprekidna $\Rightarrow$ nema V.A.

$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\frac{1}{1+\frac{1}{e^{2x}}}=1 \Rightarrow y=1$ je H.A. (kad $x\to+\infty$)

$\lim\limits_{x\to-\infty} f(x)=0 \Rightarrow y=1$ je H.A. (kad $x\to-\infty$)

f-ja nema K.A.

RAST I OPADANJE $\quad y'=\frac{2}{(e^x+e^{-x})^2}$

$y'\neq0\ \forall x\in\mathbb{R} \Rightarrow y\nearrow\ \forall x$

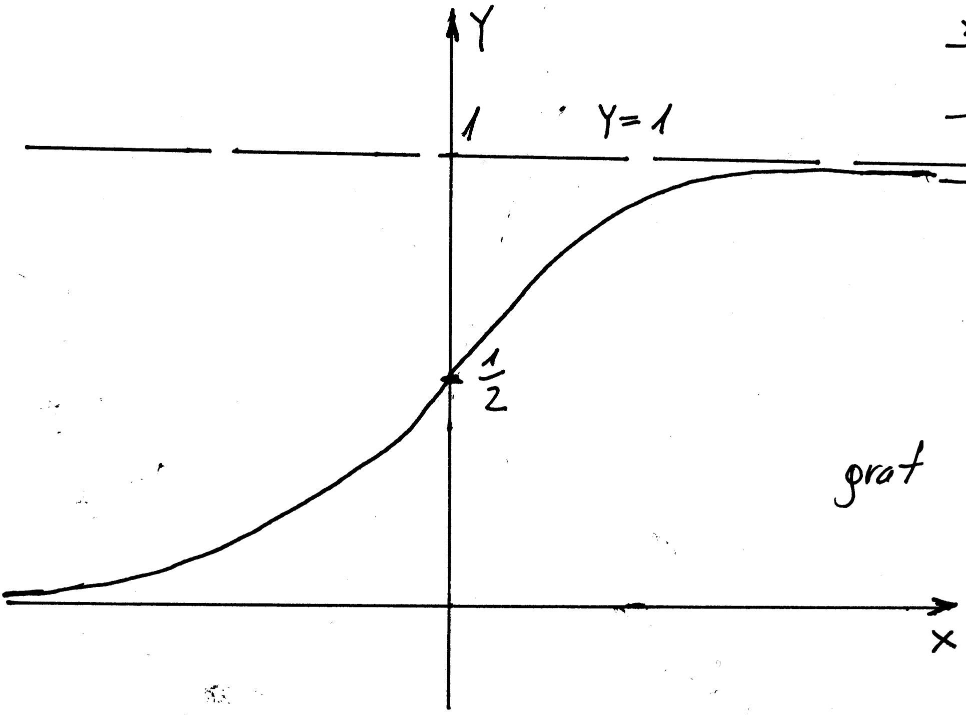
EKSTREMI

f-ja nema ekstrema

PREV. TAČK. I INT. KONV. I KONKAV.

$$y''=-4\frac{e^{2x}(e^{2x}-1)}{(e^{2x}+1)^3}=-4\frac{e^{-x}(e^{2x}-1)}{(e^x+e^{-x})^3}$$

$y''=0$ akko $x=0$



| x | $(-\infty,0)$ | $(0,+\infty)$ |
|---|---|---|
| y'' | + | − |
| y | ∪ | ∩ |

intervali konveksnosti i konkavnosti

P.T.

P.T.$(0;\frac{1}{2})$

graf f-je $y=\frac{e^x}{e^x+e^{-x}}$

(#) Ispitati f-ju i nacrtati njen grafik $Y=\frac{3x-1}{(x+1)^3}$:

Rješenje-upute:

.DEFINICIONO PODRUČJE

$x\in(-\infty,-1)\cup(-1,+\infty)$

PARNOST (NEPARNOST), PERIODIČNOST

f-ja nije ni parna ni neparna
f-ja nije periodična

NULE, PRESJEK SA Y-OSOM, ZNAK

Nula f-je je $(\frac{1}{3};0)$.

Presjek sa y-osom je $(0;-1)$

| x | $(-\infty,-1)$ | $(-1,\frac{1}{3})$ | $(\frac{1}{3},+\infty)$ |
|---|---|---|---|
| Y | + | − | + |

znak f-je

PONAŠANJE NA KRAJEVIMA INTERVALA DEFINISANOSTI I ASIMPTOTE

$\lim_{x\to-1-0} f(x)=+\infty$, $\lim_{x\to-1+0} f(x)=-\infty \Rightarrow x=-1$ je $V_0A_0$ (sa desna i sa lijeve str.)

$\lim_{x\to-\infty} f(x)=0$, $\lim_{x\to+\infty} f(x)=0 \Rightarrow Y=0$ je $H_0A_0$

Poslije ovog koraka počinjemo skicirati grafik f-je

RAST I OPADANJE

$$Y'=(-6)\frac{x-1}{(x+1)^4}$$

$$f(1)=\frac{1}{4}$$

| x | $(-\infty,-1)$ | $(-1,1)$ | $(1,+\infty)$ |
|---|---|---|---|
| $Y'$ | + | + | − |
| Y | ↗ | ↗ | ↘ |

MAX

tabela rasta i opadanj

EKSTREMI F-JE

Maksimum f-je je u tački $(1;\frac{1}{4})$.

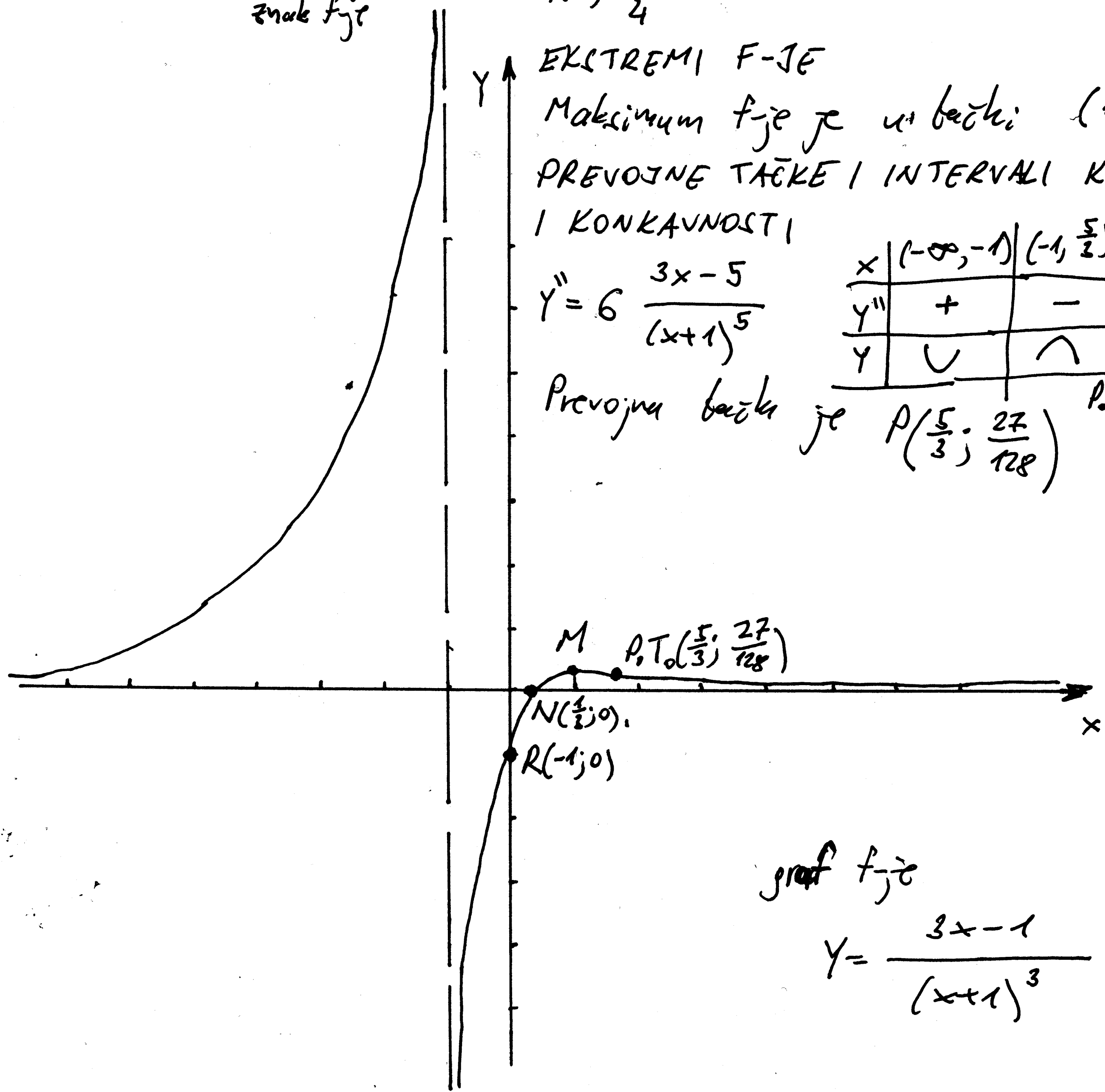
PREVOJNE TAČKE I INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$$Y''=6\frac{3x-5}{(x+1)^5}$$

| x | $(-\infty,-1)$ | $(-1,\frac{5}{3})$ | $(\frac{5}{3},+\infty)$ |
|---|---|---|---|
| $Y''$ | + | − | + |
| Y | ∪ | ∩ | ∪ |

$P_oT_o$

Prevojna tačka je $P(\frac{5}{3};\frac{27}{128})$



graf f-je

$$Y=\frac{3x-1}{(x+1)^3}$$

(#) Ispitati f-ju i nacrtati njen grafik $y = \ln \frac{2-x^2}{x}$.

Rješenje-upute:

DEFINICIONO PODRUČJE

$\mathcal{D}: x \in (-\infty, -\sqrt{2}) \cup (0, \sqrt{2})$

PARNOST (NEPARNOST), PERIODIČNOST

f-ja nije ni parna ni neparna
f-ja nije periodična

NULE, PRESJEK SA y-osom, ZNAK

Nule f-je su $M(-2;0)$ i $N(1;0)$
f-ja ne siječe y-osu

| x | $(-\infty, -2)$ | $(-2, -\sqrt{2})$ | $(0,1)$ | $(1, \sqrt{2})$ |
|---|---|---|---|---|
| y | + | − | + | − |

znak f-je

PONAŠANJE NA KRAJEVIMA INTERVALA DEFINISANOSTI I ASIMPTOTE

$\lim\limits_{x\to-\sqrt{2}+0} f(x) = -\infty$, $\lim\limits_{x\to 0+} f(x) = +\infty$,

$\lim\limits_{x\to\sqrt{2}-0} f(x) = -\infty \Rightarrow x = -\sqrt{2}$, $x = 0$ i $x = \sqrt{2}$ su V.A.

(dvije su desne i jedna sa lijeve strane)

$\lim\limits_{x\to-\infty} f(x) = \infty$, $\lim\limits_{x\to-\infty} \frac{f(x)}{x} = 0$

$\Rightarrow$ f-ja nema ni kosu ni horizontalnu asimptotu

Poslije ovog koraka počinjemo skicirati grafik f-je.

RAST I OPADANJE

$$y' = \frac{x^2+2}{x^3-2x}$$

| x | $(-\infty, -\sqrt{2})$ | $(0, \sqrt{2})$ |
|---|---|---|
| y' | − | − |
| y | ↘ | ↘ |

tabela rasta i opadanja

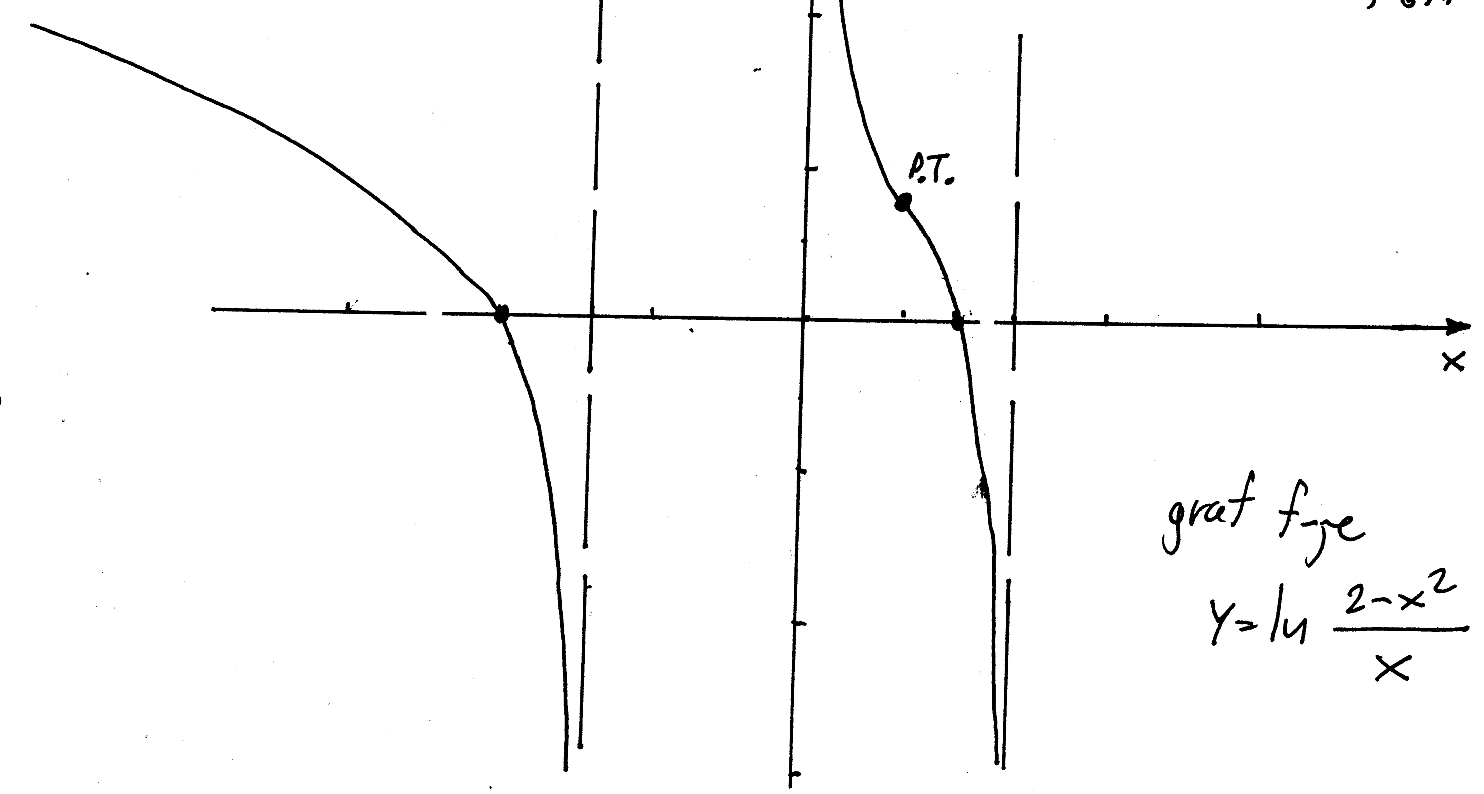
EKSTREMI

f-ja nema ekstrema

PREVOJNE TAČKE I INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$$y'' = -\frac{x^4+8x^2-4}{x^2(x^2-2)^2}$$

nule $x -0{,}6871$
$x\ 0{,}6871$



graf f-je $y = \ln \frac{2-x^2}{x}$

Ⓣ Ispitati f-ju i nacrtati njen grafik $y = \frac{e^x - e^{-x}}{e^x}$.

Rješenje-upute:

DEFINICIONO PODRUČJE

$D: x \in R$

$x \in (-\infty, +\infty)$

PARNOST (NEPARNOST), PERIODIČNOST

f-ja nije ni parna ni neparna

f-ja nije periodična

RAST I OPADANJE

$y' = 2e^{-2x} \quad (y' = 2\frac{e^{-x}}{e^x})$

$y' > 0 \; \forall x \Rightarrow$ f-ja raste za $\forall x$

EKSTREMI

f-ja nema ekstrema

NULE, PRESJEK SA y-OSOM, ZNAK

Tačka (0,0) je nula f-je i presjek sa y-osom.

| x | $(-\infty, 0$ | $(0, +\infty$ |
|---|---|---|
| y | − | + |

znak f-je

PONAŠANJE NA KRAJEVIMA INTERVALA DEFINISANOSTI I ASIMPTOTE

F-ja nema tački prekida ⇒ f-ja nema vertikalnu asimptotu

$\lim_{x\to\infty} f(x) = 1 \qquad \lim_{x\to-\infty} f(x) = -\infty$

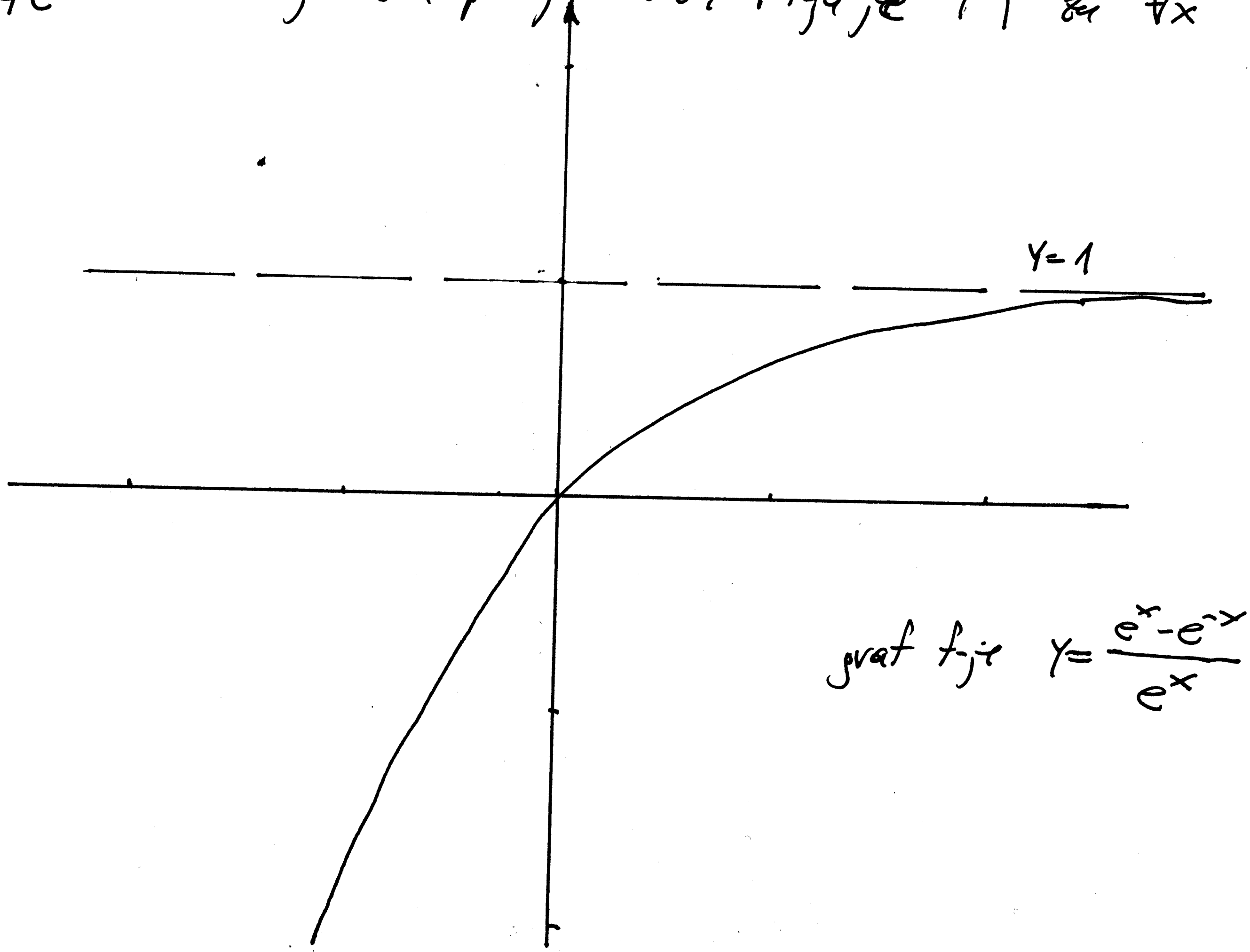
$y=1$ je H.A. kad $x \to \infty$

f-ja nema kose asimptote

Poslije ovog koraka počinjemo skicirati graf f-je.

PREVOJNE TAČKE I INTERVALI KONVEKSNOSTI I KONKAVNOSTI

$y'' = -4e^{-2x} \Rightarrow$ f-ja nema prevojnih tački i f-ja je $\cap$ za $\forall x$



graf f-je $y = \frac{e^x - e^{-x}}{e^x}$

# Odrediti stacionarne tačke f-je

$$z = \frac{1}{2}x^2 - xy + xy^2 - \frac{1}{2}x^2y$$

Rj:

$$\frac{\partial z}{\partial x} = x - y + y^2 - xy$$

$$\frac{\partial z}{\partial y} = -x + 2xy - \frac{1}{2}x^2$$

$$x - y + y^2 - xy = 0$$
$$-x + 2xy - \frac{1}{2}x^2 = 0$$

---

$$x - y + y(y - x) = 0$$
$$-x + 2xy - \frac{1}{2}x^2 = 0$$

---

$$(x-y) - y(x-y) = 0$$
$$-x + 2xy - \frac{1}{2}x^2 = 0$$

---

$$(x-y)(1-y) = 0$$
$$-x + 2xy - \frac{1}{2}x^2 = 0 \quad \dots(*)$$

---

$x - y = 0$ ili $1 - y = 0$

a) $1 - y = 0$

$y = 1$

$$(*) \Rightarrow \quad -x + 2x - \frac{1}{2}x^2 = 0$$

$$x - \frac{1}{2}x^2 = 0$$

$$x\left(1 - \frac{1}{2}x\right) = 0$$

$x = 0$ ili $x = 2$ $\qquad M_1(0,1),\ M_2(2,1)$

b)

$x - y = 0$

$$x = y \overset{(*)}{\Rightarrow} \quad -x + 2x^2 - \frac{1}{2}x^2 = 0$$

$$-x + \frac{3}{2}x^2 = 0$$

$$x\left(-1 + \frac{3}{2}x\right) = 0$$

$x = 0 \Rightarrow y = 0$

$x = \frac{2}{3} \Rightarrow y = \frac{2}{3}$

$M_3(0;0),\ M_4\left(\frac{2}{3};\frac{2}{3}\right)$

Stacionarne tačke su

$M_1(0;1),\ M_2(2;1),\ M_3(0;0),\ M_4\left(\frac{2}{3};\frac{2}{3}\right).$

(#) Odrediti stacionarne tačke f-je

$$z = 9x^2 - \frac{9}{2}x^2y + 6xy^2 - 12xy$$

Rj.

$$\frac{\partial z}{\partial x} = 18x - 9xy + 6y^2 - 12y$$

$$\frac{\partial z}{\partial y} = -\frac{9}{2}x^2 + 12xy - 12x$$

$18x - 9xy + 6y^2 - 12y = 0 \quad /:3$

$-\frac{9}{2}x^2 + 12xy - 12x = 0 \quad /:3$

---

$6x\boxed{-3xy + 2y^2} - 4y = 0$

$-\frac{3}{2}x^2 + 4xy - 4x = 0 \quad /\cdot 2$

---

$y(-3x + 2y) - 2(2y - 3x) = 0$

$-3x^2 + 8xy - 8x = 0$

---

$(y-2)(2y-3x) = 0$

$-3x^2 + 8xy - 8x = 0$

---

$y - 2 = 0$ ili $2y - 3x = 0$

a) $y - 2 = 0$

$y = 2$

$-3x^2 + 8x\cdot 2 - 8x = 0$

$8x - 3x^2 = 0 \quad /:3$

$x(8 - 3x) = 0$

$x = 0$ ili $x = \frac{8}{3}$

$M_1(0; 2)$, $M_2(\frac{8}{3}; 2)$

b) $2y - 3x = 0$

$y = \frac{3}{2}x$

$-3x^2 + \overset{4}{\cancel{8}}\cdot x \cdot \frac{3}{\underset{1}{\cancel{2}}}x - 8x = 0$

$-3x^2 + 12x^2 - 8x = 0$

$9x^2 - 8x = 0$

$x(9x - 8) = 0$

$x_1 = 0 \Rightarrow y_1 = 0$

$x_2 = \frac{8}{9} \Rightarrow$

$y_2 = \frac{3}{2}\cdot\frac{8}{9} = \frac{4}{3}$

$M_3(0; 0)$ $M_4(\frac{8}{9}; \frac{4}{3})$

Stacionarne tačke su $M_1(0; 2)$, $M_2(\frac{8}{3}; 2)$, $M_3(0; 0)$ i $M_4(\frac{8}{9}; \frac{4}{3})$

(#) Odrediti stacionarne tačke f-je

$$z = x^2y - \frac{1}{2}xy^2 - xy + \frac{1}{2}y^2$$

Rj.

$$\frac{\partial z}{\partial x} = 2xy - \frac{1}{2}y^2 - y$$

$$\frac{\partial z}{\partial y} = x^2 - xy - x + y$$

$$2xy - \frac{1}{2}y^2 - y = 0$$
$$x^2 - xy - x + y = 0$$

---

$$2xy - \frac{1}{2}y^2 - y = 0$$
$$x(x-y) - 1\cdot(x-y) = 0$$

---

$$2xy - \frac{1}{2}y^2 - y = 0 \quad \ldots (1)$$
$$(x-y)(x-1) = 0$$

---

$x - y = 0$, ili $x - 1 = 0$

a) $x - 1 = 0$

$x = 1$

$$(1) \Rightarrow \quad 2y - \frac{1}{2}y^2 - y = 0$$
$$y - \frac{1}{2}y^2 = 0$$
$$y\left(1 - \frac{1}{2}y\right) = 0$$
$$y = 0 \text{ ili } y = 2$$

$M_1(1;0)$, $M_2(1;2)$

b) $x - y = 0$

$$x = y \overset{(1)}{\Rightarrow} \quad 2x^2 - \frac{1}{2}x^2 - x = 0$$
$$\frac{3}{2}x^2 - x = 0$$
$$x\left(\frac{3}{2}x - 1\right) = 0$$

$x = 0 \Rightarrow y = 0$

$x = \frac{2}{3} \Rightarrow y = \frac{2}{3}$

$M_3(0;0)$, $M_4\left(\frac{2}{3};\frac{2}{3}\right)$

Stacionarne tačke su $M_1(1;0)$, $M_2(1;2)$, $M_3(0;0)$ i $M_4\left(\frac{2}{3};\frac{2}{3}\right)$.

(#) Odrediti stacionarne tačke f-je

$$z = 6x^2y - \frac{9}{2}xy^2 - 12xy + 9y^2$$

Rj.

$$\frac{\partial z}{\partial x} = 12xy - \frac{9}{2}y^2 - 12y$$

$$\frac{\partial z}{\partial y} = 6x^2 - 9xy - 12x + 18y$$

$$12xy - \frac{9}{2}y^2 - 12y = 0 \quad /:3$$
$$6x^2 - 9xy - 12x + 18y = 0 \quad /:3$$

---

$$4xy - \frac{3}{2}y^2 - 4y = 0 \quad /\cdot 2$$
$$2x^2 - 3xy - 4x + 6y = 0$$

---

$$8xy - 3y^2 - 8y = 0$$
$$x(2x - 3y) - 2(2x - 3y) = 0$$

---

$$8xy - 3y^2 - 8y = 0$$
$$(x-2)(2x-3y) = 0$$

---

$$x - 2 = 0 \quad \text{ili} \quad 2x - 3y = 0$$

a)

$$x - 2 = 0$$
$$x = 2$$
$$8 \cdot 2 \cdot y - 3y^2 - 8y = 0$$
$$8y - 3y^2 = 0$$
$$y(8 - 3y) = 0$$
$$y = 0 \quad \text{ili} \quad y = \frac{8}{3}$$

$M_1(2; 0)$, $M_2(2; \frac{8}{3})$

b)

$$2x - 3y = 0$$
$$x = \frac{3}{2}y \qquad \overset{4}{\cancel{8}} \cdot \frac{3}{\cancel{2}_1} y \cdot y - 3y^2 - 8y = 0$$
$$-8y + 9y^2 = 0$$
$$y(9y - 8) = 0$$

$$y_1 = 0 \Rightarrow x_1 = 0$$
$$y_2 = \frac{8}{9} \Rightarrow x_2 = \frac{3}{2} \cdot \frac{8}{9} = \frac{4}{3}$$

$M_3(0; 0)$, $M_4(\frac{4}{3}; \frac{8}{9})$

Stacionarne tačke f-je su

$M_1(2; 0)$, $M_2(2; \frac{8}{3})$, $M_3(0; 0)$, $M_4(\frac{4}{3}; \frac{8}{9})$

ⓘ# Nađi ekstreme f-je $z=x^3+3xy^2-15x-12y$.

Rj.

$$\frac{\partial z}{\partial x}=3x^2+3y^2-15$$

$$\frac{\partial z}{\partial y}=6xy-12$$

$3x^2+3y^2-15=0 \quad /:3$

$6xy-12=0 \quad /:6$

---

$x^2+y^2-5=0 \quad /\cdot x^2$

$xy-2=0 \Rightarrow xy=2$

---

$x^4+(xy)^2-5x^2=0$

$x^4-5x^2+4=0$

$x^2=t \Rightarrow t^2-5t+4=0$

$D=25-16=9$

$$t_{1,2}=\frac{5\pm 3}{2}$$

$$t_1=\frac{2}{2}=1$$

$$t_2=\frac{8}{2}=4$$

$t_1=1 \Rightarrow x^2=1$

$x_1=-1 \Rightarrow -y=2 \Rightarrow y_1=-2$

$x_2=1 \Rightarrow y=2 \Rightarrow y_2=2$

$t_2=4 \Rightarrow x^2=4$

$x_3=-2 \Rightarrow -2y=2 \Rightarrow y=-1$

$x_4=2 \Rightarrow 2y=2 \Rightarrow y=1$

Stacionarne tačke su $M_1(-1,-2)$, $M_2(1,2)$, $M_3(-2,-1)$ ; $M_4(2,1)$.

$$\frac{\partial^2 z}{\partial x^2}=6x$$

$$\frac{\partial^2 z}{\partial x\partial y}=6y$$

$$\frac{\partial^2 z}{\partial y^2}=6x$$

$M_1(-1,-2)$: $A=-6$, $B=-12$, $C=-6$

$D=AC-B^2=36-12^2=36-144<0$

f-ja u tački $M_1$ nema ekstrema

$M_2(1,2)$: $A=6$, $B=12$, $C=6$

$D=AC-B^2=36-144<0$ f-ja u tački $M_2$ nema ekstrema

$M_3(-2,-1)$: $A=-12$, $B=-6$, $C=-12$, $D=AC-B^2=12^2-6^2>0$

f-ja u tački $M_3$ ima ekstrem, $A<0 \Rightarrow$ f-ja ima max

$z_{max}(-2,-1)=-8-6+30+12=42-14=28$

$M_4(2,1)$: $A=12$, $B=6$, $C=12$, $D=AC-B^2=12^2-6^2>0$

f-ja u tački $M_3$ ima ekstrem, $A>0 \Rightarrow$ f-ja ima min

$z_{min}(2,1)=8+6-30-12=14-42=-28$

(#) Odrediti ekstreme f-je $z = x^2 + y^3 + 4x\sqrt{x^3} - 3y$.

Rj.-upute

$$\frac{\partial z}{\partial x} = 2x + 10\sqrt{x^3}$$

$$2x + 10x\sqrt{x} = 0$$

$$\frac{\partial z}{\partial y} = 3y^2 - 3$$

$$3y^2 - 3 = 0$$

⋮

Stacionarne tačke sa

$M(0;1)$ i $N(0;-1)$

$$\frac{\partial^2 z}{\partial x^2} = 15\sqrt{x} + 2$$

$$\frac{\partial^2 z}{\partial x \partial y} = 0$$

$$\frac{\partial^2 z}{\partial y^2} = 6y$$

Za $M(0;1)$

$$D = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = 12$$

$z_{MAX}(0;1) =$

Za $N(0;-1)$ imamo $D = -12$

$D < 0$

U tački N f-ja nema ekstrem

#) Odrediti ekstreme f-je $z = 3\ln\frac{x}{6} + \ln(12-y-x) + 2\ln y$

Rj.-upute

$$\frac{\partial z}{\partial x} = \frac{1}{x+y-12} + \frac{3}{x} \qquad \frac{1}{x+y-12} + \frac{3}{x} = 0$$

$$\frac{\partial z}{\partial y} = \frac{1}{x+y-12} + \frac{2}{y} \qquad \frac{1}{x+y-12} + \frac{2}{y} = 0$$

$$\vdots$$

Stacionarna tačka je $M(6;4)$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{(x+y-12)^2} - \frac{3}{x^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{(x+y-12)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{(x+y-12)^2} - \frac{2}{y^2}$$

Za $M(6;4)$ imamo

$A = -\frac{1}{3}$

$B = -\frac{1}{4}$

$C = -3/8$

$$D = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = \frac{1}{16} > 0$$

$$z_{MAX}(6;4) = 3\ln 1 + \ln 2 + 2\ln 4 = \ln 2 + 2\ln 4$$

$$= \ln 2 + \ln 4^2 = \ln 32$$

(#) Odrediti ekstreme f-je $z=x^3+y^2-3x+4\sqrt{y^5}$.

Rj.-upute

$$\frac{\partial z}{\partial x}=3x^2-3$$

$$\frac{\partial z}{\partial y}=2y+\frac{10y^4}{\sqrt{y^5}}=2y+\frac{10y^2}{\sqrt{y}}=2y+10y\sqrt{y}=2y+10\sqrt{y^3}$$

$$3x^2-3=0$$
$$y(2+10\sqrt{y})=0$$

---

$\vdots$

Stacionarne tačke su

$M(1;0)$ i $N(-1;0)$

$$\frac{\partial^2 z}{\partial x^2}=6x$$

$$\frac{\partial^2 z}{\partial x\partial y}=0$$

$$\frac{\partial^2 z}{\partial y^2}=15\sqrt{y}+2$$

Za $M(1;0)$

$$D=\begin{vmatrix} A & B \\ B & C \end{vmatrix}=12$$

– $z_{MAX}(1;0)=-2$

Za $N(-1;0)$

$$D=-12<0$$

U tački N f-ja nema ekstrema.

(#) Odrediti ekstreme f-je $z = 2\ln x + \ln(12-x-y) + 3\ln\frac{y}{6}$

Rj.-upute

$$\frac{\partial z}{\partial x} = \frac{1}{x+y-12} + \frac{2}{x} \qquad\qquad \frac{1}{x+y-12} + \frac{2}{x} = 0$$

$$\frac{\partial z}{\partial y} = \frac{1}{x+y-12} + \frac{3}{y} \qquad\qquad \underline{\frac{1}{x+y-12} + \frac{3}{y} = 0}$$

$$\vdots$$

Stacionarna tačka je $N(4;6)$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{(x+y-12)^2} - \frac{2}{x^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{(x+y-12)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{(x+y-12)^2} - \frac{3}{y^2}$$

Za $N(4;6)$

$A = -\frac{3}{8}$, $B = -\frac{1}{4}$, $C = -\frac{1}{3}$

$D = \frac{1}{16} > 0$

$$z_{MAX}(4;6) = \ln 32$$

**#** Naći ekstreme f-je $z=\frac{1}{3}x^3-2xy+x+3y^2-4y$.

Rj.-upute

$z'_x=x^2-2y+1$

$z'_y=-2x+6y-4$

$$\begin{array}{l} x^2-2y+1=0 \quad /\cdot 3 \\ \underline{-2x+6y-4=0} \\ 3x^2-6y+3=0 \\ -2x+6y-4=0 \end{array}$$

$\vdots$

Stacionarne tačke su $M_1(1;1)$ i $M_2\left(-\frac{1}{3};\frac{5}{9}\right)$

$\frac{\partial^2 z}{\partial x^2}=2x$

$\frac{\partial^2 z}{\partial x \partial y}=-2$

$\frac{\partial^2 z}{\partial y^2}=6$

$M_1(1;1)$

$A=2,\ B=-2,\ C=6$

$D=8>0 \Rightarrow$ f-ja ima ekstrem

$A>0 \Rightarrow$ f-ja ima minimum

$z_{min}(1;1)=\ldots=-\frac{5}{3}$

$M_2\left(-\frac{1}{3};\frac{5}{9}\right)$

$\vdots$

$D=-8<0 \Rightarrow$ f-ja u tački $M_2$ nema ekstrem

(#) Izračunati integral $I=\int \frac{\sin x \cdot \cos x}{e^x}\,dx.$

Rj.

$$I=\int \frac{\sin x \cdot \cos x}{e^x}\,dx = \frac{1}{2}\int \frac{2\sin x\cos x}{e^x}\,dx = \frac{1}{2}\int \frac{\sin 2x}{e^x}\,dx =$$

$$= \frac{1}{2}\int e^{-x}\sin 2x\,dx = \begin{vmatrix} u=e^{-x} & dv=\sin 2x\,dx \\ du=-e^{-x}dx & v=-\frac{1}{2}\cos 2x \end{vmatrix} =$$

$$= \frac{1}{2}\left(-\frac{1}{2}\,e^{-x}\cos 2x - \frac{1}{2}\int e^{-x}\cos 2x\,dx\right) =$$

$$= -\frac{1}{4}\,e^{-x}\cos 2x - \frac{1}{4}\int e^{-x}\cos 2x\,dx$$

$$\int e^{-x}\cos 2x\,dx = \begin{vmatrix} u=e^{-x} & dv=\cos 2x\,dx \\ du=-e^{-x}dx & v=\frac{1}{2}\sin 2x \end{vmatrix} = \frac{1}{2}e^{-x}\sin 2x + \frac{1}{2}\int e^{-x}\sin 2x\,dx$$

Dobili smo

$$\underline{\frac{1}{2}\int e^{-x}\sin 2x\,dx} = -\frac{1}{4}e^{-x}\cos 2x - \frac{1}{8}\,e^{-x}\sin 2x \underline{- \frac{1}{8}\int e^{-x}\sin 2x\,dx}$$

$$\frac{5}{8}\int e^{-x}\sin 2x\,dx = -\frac{1}{4}e^{-x}\cos 2x - \frac{1}{8}\,e^{-x}\sin 2x \quad /\cdot\frac{8}{10}$$

$$\frac{1}{2}\int e^{-x}\sin 2x\,dx = -\frac{2}{10}\,e^{-x}\cos 2x - \frac{1}{10}\,e^{-x}\sin 2x$$

Kako je $I=\int \frac{\sin x \cdot \cos x}{e^x}\,dx = \frac{1}{2}\int e^{-x}\sin 2x\,dx$ to je

$$\int \frac{\sin x \cdot \cos x}{e^x}\,dx = -\frac{1}{10}\,e^{-x}\left(2\cos 2x + \sin 2x\right)$$

㊉ Izračunati integral $\int \frac{\sin x+\cos x}{\sin x+2\cos x}\,dx$

Rj.: $\int \frac{\sin x+\cos x}{\sin x+2\cos x}\begin{matrix}/:\cos x\\ /:\cos x\end{matrix}\,dx = \int \frac{\mathrm{tg}\,x+1}{\mathrm{tg}\,x+2}\,dx = \left|\begin{matrix}\mathrm{tg}\,x=t\\ x=\mathrm{arctg}\,t\\ dx=\frac{dt}{1+t^2}\end{matrix}\right| =$

$$= \int \frac{t+1}{t+2}\cdot\frac{1}{1+t^2}\,dt = \int \frac{t+1}{(t+2)(1+t^2)}\,dt$$

$$\frac{t+1}{(t+2)(t^2+1)} = \frac{A}{t+2} + \frac{Bt+C}{t^2+1} \quad /(t+2)(t^2+1)$$

$$t+1 = A(t^2+1)+(Bt+C)(t+2)$$

$$t+1 = A(t^2+1)+B(t^2+2t)+C(t+2)$$

$A+B=0 \Rightarrow A=-B$

$2B+C=1$

$A+2C=1 \Rightarrow A=1-2C$

$A=-B$

$A=1-2C$

$-B=1-2C \quad /(-1)$

$B=2C-1$

$2B+C=1$

$2(2C-1)+C=1$

$4C-2+C=1$

$5C=3$

$C=\frac{3}{5}$, $A=1-\frac{6}{5}=-\frac{1}{5}$

$B=\frac{1}{5}$

$$\int\frac{t+1}{(t+2)(t^2+1)}\,dt = \int\frac{-\frac{1}{5}}{t+2}\,dt + \int\frac{\frac{1}{5}t+\frac{3}{5}}{t^2+1}\,dt = -\frac{1}{5}\ln|t+2| + \frac{1}{5}\int\frac{t+3}{t^2+1}\,dt$$

$$= -\frac{1}{5}\ln|t+2| + \frac{1}{5}\int\frac{t\,dt}{t^2+1} + \frac{3}{5}\int\frac{dt}{t^2+1} = \left|\begin{matrix}t^2+1=s\\ 2t\,dt=ds\\ t\,dt=\frac{ds}{2}\end{matrix}\right| =$$

$$= -\frac{1}{5}\ln|t+2| + \frac{1}{10}\ln|s| + \frac{3}{5}\,\mathrm{arctg}\,t + C =$$

$$= -\frac{1}{5}\ln|\mathrm{tg}\,x+2| + \frac{1}{10}\ln|\mathrm{tg}^2x+1| + \frac{3}{5}\,\mathrm{arctg}\,(\mathrm{tg}\,x) + C$$

(#) Odrediti sljedeće integrale

a) $\int (x^2+2x)\cos 2x \, dx$ b) $\int \left(\frac{3}{2}x^2+3x\right)\sin 3x \, dx$

c) $\int x \operatorname{arctg} x \, dx$ d) $\int x \operatorname{arcctg} x \, dx$

Rj.

a)
$$\int (x^2+2x)\cos 2x \, dx = \left| \begin{matrix} u = x^2+2x & dv = \cos 2x \, dx = \frac{1}{2}\cos 2x \, d(2x) \\ du = (2x+2)\,dx & v = \frac{1}{2}\sin 2x \end{matrix} \right|$$

$$= \frac{1}{2}(x^2+2x)\sin 2x - \int (x+1)\sin 2x \, dx = \left| \begin{matrix} u = x+1 & dv = \sin 2x \, dx \\ du = dx & v = -\frac{1}{2}\cos 2x \end{matrix} \right|$$

$$= \frac{1}{2}(x^2+2x)\sin 2x - \left( -\frac{1}{2}(x+1)\cos 2X + \frac{1}{2}\int \cos 2x \, dx \right) =$$

$$= \frac{1}{2}(x^2+2x)\sin 2x + \frac{1}{2}(x+1)\cos 2x - \frac{1}{4}\sin 2x + C$$

b)
$$\int \left(\frac{3}{2}x^2+3x\right)\sin 3x \, dx = \left| \begin{matrix} u = \frac{3}{2}x^2+3x & dv = \sin 3x \, dx = \frac{1}{3}\sin 3x \, d(3x) \\ du = 3x+3 & v = -\frac{1}{3}\cos 3x \end{matrix} \right|$$

$$= \left(-\frac{1}{2}x^2 - x\right)\cos 3x + \int (x+1)\cos 3x \, dx = \left| \begin{matrix} u = x+1 & dv = \cos 3x \, dx \\ du = dx & v = \frac{1}{3}\sin 3x \end{matrix} \right|$$

$$= (-1)\left(\frac{x^2}{2}+x\right)\cos 3x + \frac{1}{3}(x+1)\sin 3x - \frac{1}{3}\int \sin 3x \, dx =$$

$$= -\frac{1}{2}(x^2+2x)\cos 3x + \frac{1}{3}(x+1)\sin 3x + \frac{1}{9}\cos 3x + C$$

c) $$\int x \operatorname{arc\,tg} x \, dx = \left| \begin{array}{ll} u = \operatorname{arc\,tg} x & dv = x\,dx \\ du = \dfrac{dx}{1+x^2} & v = \dfrac{x^2}{2} \end{array} \right| = \frac{x^2}{2} \operatorname{arc\,tg} x - \frac{1}{2} \int \frac{\overset{+1-1}{x^2}}{1+x^2} dx$$

$$= \frac{x^2}{2} \operatorname{arc\,tg} x - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{dx}{1+x^2} =$$

$$= \frac{1}{2} x^2 \operatorname{arc\,tg} x - \frac{1}{2} x + \frac{1}{2} \operatorname{arc\,tg} x + C$$

d) $$\int x \operatorname{arc\,ctg} x \, dx = \left| \begin{array}{ll} u = \operatorname{arc\,ctg} x & dv = x\,dx \\ du = \dfrac{-1}{1+x^2} & v = \dfrac{x^2}{2} \end{array} \right| =$$

$$= \frac{x^2}{2} \operatorname{arc\,ctg} x + \frac{1}{2} \int \frac{x^2 \; {}^{+1-1}}{1+x^2} dx =$$

$$= \frac{x^2}{2} \operatorname{arc\,ctg} x + \frac{1}{2} \int dx - \frac{1}{2} \int \frac{dx}{1+x^2} =$$

$$= \frac{x^2}{2} \operatorname{arc\,ctg} x + \frac{1}{2} x - \frac{1}{2} \operatorname{arc\,tg} x + C$$

# ⑪ Odrediti integrale

a) $\int \frac{5x-3}{\sqrt{-34+12x-x^2}} d_x$ c) $\int \frac{2x-1}{\sqrt{-7+6x-x^2}} d_x$

b) $\int \frac{4x+2}{\sqrt{-22+10x-x^2}} d_x$ d) $\int \frac{3x-7}{\sqrt{-33-12x-x^2}} d_x$

Rj. JEDAN OD NAČINA ZA RJEŠAVANJE

a)

$$-34+12x-x^2 = -(x^2-12x+34) = -(x^2-2\cdot x\cdot 6+6^2-6^2+34) =$$

$$= -((x-6)^2-2) = 2-(x-6)^2$$

$$\int \frac{5x-3}{\sqrt{-34+12x-x^2}} d_x = \int \frac{5x-3^{\,-27+27}}{\sqrt{2-(x-6)^2}} d_x = \left| \begin{array}{l} \text{želimo uvesti smjenu} \\ 2-(x-6)^2 = s \\ -2(x-6)\,d_x = d_s \\ (x-6)\,d_x = -\frac{1}{2} d_s \\ 5(x-6)\,d_x = -\frac{5}{2} d_s \\ (5x-30)\,d_x = -\frac{5}{2} d_s \end{array} \right|$$

$$= 5\int \frac{(x-6)\,d_x}{\sqrt{2-(x-6)^2}} d_x + 27\int \frac{d_x}{\sqrt{2-(x-6)^2}} =$$

$$= \left| \begin{array}{l} \text{uvodimo smjenu} \\ 2-(x-6)^2 = s \\ (x-6)\,d_x = -\frac{1}{2} d_s \end{array} \right| = 5\cdot\left(-\frac{1}{2}\right)\int s^{-\frac{1}{2}} d_s + 27\int \frac{d(x-6)}{\sqrt{2-(x-6)^2}} =$$

$$= -\frac{5}{2}\cdot 2\cdot s^{\frac{1}{2}} + 27 \arcsin \frac{x-6}{\sqrt{2}} + C = -5\sqrt{2-(x-6)^2} + 27 \arcsin \frac{x-6}{\sqrt{2}} + C$$

b)

$$-22+10x-x^2 = -(x^2-10x+22) = -(x^2-2\cdot x\cdot 5+5^2-5^2+22) =$$

$$= -((x-5)^2-3) = 3-(x-5)^2$$

$$\int \frac{4x+2}{\sqrt{-22+10x-x^2}} d_x = \int \frac{4x+2^{\,-22+22}}{\sqrt{3-(x-5)^2}} d_x = \left| \begin{array}{l} \text{želimo uvesti smjenu} \\ 3-(x-5)^2 = s \\ -2(x-5)\,d_x = d_s \quad (x-5)\,d_x = -\frac{1}{2} d_s \end{array} \right|$$

$$= 4\int \frac{(x-5)\;dx}{\sqrt{3-(x-5)^2}}\,dx + 22\int \frac{dx}{\sqrt{3-(x-5)^2}} = \left|\begin{array}{l} \text{uvedimo smjenu} \\ 3-(x-5)^2 = s \\ (x-5)dx = -\frac{1}{2}ds \end{array}\right|$$

$$= 4\cdot\left(-\frac{1}{2}\right)\int s^{-\frac{1}{2}}ds + 22\int \frac{d(x-5)}{\sqrt{3-(x-5)^2}} = (-2)\cdot 2\, s^{\frac{1}{2}} + 22\arcsin\frac{x-5}{\sqrt{3}} + C$$

$$= -4\sqrt{3-(x-5)^2} + 22\arcsin\frac{x-5}{\sqrt{3}} + C$$ traženo rješenje

c) $-7+6x-x^2 = -(x^2-6x+7) = -(x^2-2\cdot x\cdot 3+3^2-3^2+7) =$

$= -((x-3)^2-2) = 2-(x-3)^2$

$$\int \frac{2x-1}{\sqrt{-7+6x-x^2}}dx = \int \frac{2x-1\;{}^{-5+5}}{\sqrt{2-(x-3)^2}}dx = \left|\begin{array}{l} \text{želimo uvesti smjenu} \\ 2-(x-3)^2 = s \\ -2(x-3)\,dx = ds \quad (x-3)dx = -\frac{1}{2}ds \end{array}\right|$$

$$= 2\int \frac{(x-3)\,dx}{\sqrt{2-(x-3)^2}} + 5\int \frac{dx}{\sqrt{2-(x-3)^2}} = \left|\begin{array}{l} 2-(x-3)^2 = s \\ (x-3)dx = -\frac{1}{2}ds \end{array}\right| =$$

$$= 2\cdot\left(-\frac{1}{2}\right)\int s^{-\frac{1}{2}}ds + 5\int \frac{d(x-3)}{\sqrt{2-(x-3)^2}} = (-1)\cdot 2\, s^{\frac{1}{2}} + 5\arcsin\frac{x-3}{\sqrt{2}} + C$$

$$= -2\sqrt{2-(x-3)^2} + 5\arcsin\frac{x-3}{\sqrt{2}} + C$$ traženo rješenje

d) za vježbu

uputa: $-33-12x-x^2 = 3-(x+6)^2$

rješenje $I = -3\sqrt{3-(x+6)^2} - 25\arcsin\frac{x+6}{\sqrt{3}} + C$

(#) Odrediti sljedeće integrale

a) $\displaystyle\int \frac{\sqrt[6]{x+1}\,dx}{\sqrt{x+1}+\sqrt[3]{x+1}}$ b) $\displaystyle\int \frac{dx}{3x-4\sqrt{x}}$

c) $\displaystyle\int \frac{\sqrt{x}\,dx}{\sqrt[4]{x^3}+4}$ d) $\displaystyle\int \frac{\sqrt{x}\,dx}{\sqrt[3]{x^2}-\sqrt[4]{x}}$

Rj.:

a) $$\int \frac{\sqrt[6]{x+1}}{\sqrt{x+1}+\sqrt[3]{x+1}}\,dx = \left| \begin{matrix} NZS(1,3,6)=6 \\ x+1=u^6 \\ dx=6u^5du \end{matrix} \right| = \int \frac{u}{u^3+u^2}\,6u^5\,du =$$

$$=6\int \frac{u^4}{u+1}\,du \overset{(*)}{=} 6\int \left(u^3-u^2+u-1+\frac{1}{u+1}\right)du \overset{(1)}{=}$$

$u^4:(u+1)=u^3-u^2+u-1$

$-\ u^4+u^3$

$-u^3$

$-\ -u^3-u^2$

$u^2$

$-\ u^2+u$

$-u$

$-\ -u-1$

$1$

$$\frac{u^4}{u+1}=u^3-u^2+u-1+\frac{1}{u+1}$$

...(*)

$$\overset{(1)}{=} \frac{6}{4}u^4-\frac{6}{3}u^3+\frac{6}{2}u^2-6u+6\ln|u+1|+C=$$

$$=\frac{3}{2}\sqrt[3]{(x+1)^2}-2\sqrt{x+1}+3\sqrt[3]{x+1}-\sqrt[6]{x+1}+6\ln\left|\sqrt[6]{x+1}+1\right|+C$$

b) $$\int \frac{dx}{3x-4\sqrt{x}} = \left| \begin{matrix} x=u^2 \\ dx=2u\,du \end{matrix} \right| = \int \frac{2u\,du}{\underbrace{3u^2-4u}_{u(3u-4)}} = \frac{2}{3}\int \frac{du}{u-\frac{4}{3}} =$$

$$\frac{2}{3}\int\frac{d(u-\frac{4}{3})}{u-\frac{4}{3}}=\frac{2}{3}\ln\left|u-\frac{4}{3}\right|+C_1=\frac{2}{3}\ln\left|\sqrt{x}-\frac{4}{3}\right|+C_1=$$

$$\frac{2}{3}\ln\left|\frac{3\sqrt{x}-4}{3}\right|+C_1=\frac{2}{3}\ln\left|3\sqrt{x}-4\right|+C$$

c) $$\int\frac{\sqrt{x}\,dx}{\sqrt[4]{x^3}+4}=\int\frac{x^{\frac{1}{2}}}{x^{\frac{3}{4}}+4}\,dx=\left|\begin{matrix}NZS(34)=4\\ x=u^4\\ dx=4u^3du\end{matrix}\right|=$$

$$=4\int\frac{u^2}{u^3+4}u^3du=4\int\frac{u^5}{u^3+4}=4\int\left(u^2-\frac{4u^2}{u^3+4}\right)du=$$

$$u^5:(u^3+4)=u^2$$
$$\underline{u^5+4u^2}$$
$$-4u^2$$

$$=4\int u^2du-16\cdot\frac{1}{3}\int\frac{d(u^3+4)}{u^3+4}=$$

$$=\frac{4}{3}u^3-\frac{16}{3}\ln|u^3+4|+C=$$

$$=\frac{4}{3}\sqrt[4]{x^3}-\frac{16}{3}\ln\left|\sqrt[4]{x^3}+4\right|+C$$

d) $$\int\frac{\sqrt{x}\,dx}{\sqrt[3]{x^2}-\sqrt[4]{x}}=\left|\begin{matrix}NZS(2,3,4)=12\\ x=u^{12}\\ dx=12u^{11}du\end{matrix}\right|=\int\frac{u^6}{u^8-u^3}12u^{11}du=$$

$$=12\int\frac{u^{14}}{u^5-1}du=12\int\left(u^9+u^4+\frac{u^4}{u^5-1}\right)du=$$

$$u^{14}:(u^5-1)=u^9+u^4$$
$$\underline{u^{14}-u^9}$$
$$u^9$$
$$\underline{-\,u^9-u^4}$$
$$u^4$$

$$=12\cdot\frac{u^{10}}{10}+12\frac{u^5}{5}+12\cdot\frac{1}{5}\int\frac{d(u^5-1)}{u^5-1}=$$

$$=\frac{6}{5}\left(x^{\frac{1}{12}}\right)^{10}+\frac{12}{5}\left(x^{\frac{1}{12}}\right)^5+\frac{12}{5}\ln\left|\left(x^{\frac{1}{12}}\right)^5-1\right|+C$$

$$=\frac{6}{5}\sqrt[6]{x^5}+\frac{12}{5}\sqrt[12]{x^5}+\frac{12}{5}\ln\left|\sqrt[12]{x^5}-1\right|+C.$$

Ⓣ Izračunati sljedeće integrale

a) $\int x\ln(x-1)\,dx$ c) $\int \ln(x^2-1)\,dx$

b) $\int \ln(1+x^2)\,dx$ d) $\int (x+1)\ln x\,dx$

Rj.

a) $$\int x\ln(x-1)\,dx = \left| \begin{matrix} u=\ln(x-1) & dv = x\,dx \\ du = \frac{1}{x-1}dx & v=\frac{1}{2}x^2 \end{matrix} \right| = \frac{1}{2}x^2\ln(x-1) -$$

$$-\frac{1}{2}\int \frac{x^2}{x-1}\,dx = \left| \frac{x^2}{x-1} = \frac{x^2-1+1}{x-1} = \frac{(x-1)(x+1)+1}{x-1} = x+1+\frac{1}{x-1} \right|$$

$$= \frac{1}{2}x^2\ln(x-1) - \frac{1}{2}\left(\frac{1}{2}x^2 + x + \ln|x-1|\right) + C = \frac{x^2-1}{2}\ln|x-1| - \frac{x^2}{4} - \frac{x}{2} + C$$

b) $$\int \ln(1+x^2)\,dx = \left| \begin{matrix} u=\ln(1+x^2) & dv=dx \\ du = \frac{1}{1+x^2}\cdot 2x\,dx & v=x \end{matrix} \right| =$$

$$= x\ln(1+x^2) - 2\int \frac{x^2 \overset{+1-1}{}}{1+x^2}\,dx =$$

$$= x\ln(1+x^2) - 2\int \left(1-\frac{1}{1+x^2}\right)dx =$$

$$= x\ln(1+x^2) - 2x + 2\,\mathrm{arctg}\,x + C$$

c)

$$\int \ln(x^2-1)\,dx = \left| \begin{matrix} u=\ln(x^2-1) & dv=dx \\ du=\dfrac{2x}{x^2-1} & v=x \end{matrix} \right| = x\ln(x^2-1)-2\int \frac{x^2-1+1}{x^2-1}\,dx$$

$$= x\ln(x^2-1)-2\cdot\int\left(1+\frac{1}{x^2-1}\right)dx =$$

$$= x\ln(x^2-1)-2x-2\cdot\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$$

$$= x\ln(x^2-1)-2x-\ln\left|\frac{x-1}{x+1}\right|+C$$

d)

$$\int (x+1)\ln x\,dx = \left| \begin{matrix} u=\ln x & dv=x+1 \\ du=\dfrac{dx}{x} & v=\frac{1}{2}x^2+x \end{matrix} \right| =$$

$$= \left(\frac{1}{2}x^2+x\right)\ln x-\int\left(\frac{1}{2}x+1\right)dx$$

$$= \left(\frac{1}{2}x^2+x\right)\ln x-\frac{1}{4}x^2-x+C$$

## ⧣ Odrediti integrale

(a) $\int \frac{7x-17}{x^2-5x+6}\,dx$ . (b) $\int \frac{9x-2}{x^2-x-6}\,dx$

(c) $\int \frac{11x+14}{x^2+3x-4}\,dx$

Rj.-upute

(a) $\frac{7x-17}{x^2-5x+6} = \dots = \frac{3}{x-2} + \frac{4}{x-3}$

$$\int \frac{7x-17}{x^2-5x+6}\,dx = \int \frac{3}{x-2}\,dx + \int \frac{4}{x-3}\,dx = 3\ln|x-2| + 4\ln|x-3| + C$$

(b)

$$\frac{9x-2}{x^2-x-6} = \dots = \frac{4}{x+2} + \frac{5}{x-3}$$

$$\int \frac{9x-2}{x^2-x-6}\,dx = \int \frac{4}{x+2}\,dx + \int \frac{5}{x-3}\,dx = 4\ln|x+2| + 5\ln|x-3| + C$$

c)

$$\frac{11x+14}{x^2+3x-4} = \dots = \frac{5}{x-1} + \frac{6}{x+4}$$

$$\int \frac{11x+14}{x^2+3x-4}\,dx = 5\int \frac{dx}{x-1} + 6\int \frac{dx}{x+4} = 5\ln|x-1| + 6\ln|x+4| + C$$

\# Izračunati integrale

a) $\displaystyle\int \frac{x-1}{\sqrt{-1+4x-x^2}}\,dx$  (b) $\displaystyle\int \frac{4x^2+11x-2}{x^3-3x-2}\,dx$

Rj: Jedan od načina za rješavanje je sljedeći

(a) $-1+4x-x^2=-(x^2-4x+1)=-(x^2-2\cdot x\cdot 2+4-4+1)=$

$$=-((x-2)^2-3)=3-(x-2)^2$$

Prema tome

$$\int \frac{x-1}{\sqrt{-1+4x-x^2}}\,dx=\int \frac{\overset{-1+1}{x-1}}{\sqrt{3-(x-2)^2}}\,dx=\int \frac{x-2}{\sqrt{3-(x-2)^2}}\,dx+\int \frac{dx}{\sqrt{3-(x-2)^2}}$$

$$=-\int \left(3-(x-2)^2\right)^{-\frac{1}{2}}\cdot\frac{1}{2}\,d\left(3-(x-2)^2\right)+\int \frac{d(x-2)}{\sqrt{3-(x-2)^2}}=$$

$$=-\frac{1}{2}\cdot\frac{\left(3-(x-2)^2\right)^{\frac{1}{2}}}{\frac{1}{2}}+\arcsin\frac{x-2}{\sqrt{3}}+C$$

$$=-\sqrt{3-(x-2)^2}+\arcsin\frac{x-2}{\sqrt{3}}+C$$

(b) $$\frac{4x^3+11x\cdot 2}{x^3-3x-2}=\ldots=\frac{3}{(x+1)^2}+\frac{4}{x-2}$$

$$\int \frac{4x^2+11x-2}{x^3-3x-2}\,dx=\int\left(\frac{3}{(x+1)^2}+\frac{4}{x-2}\right)dx=3\int\frac{d(x+1)}{(x+1)^{-2}}+4\int\frac{d(x-2)}{x-2}$$

$$=\frac{-3}{x+1}+4\ln|x-2|$$

## (#) Odrediti integrale

(a) $\displaystyle\int \frac{6x^2-19x+9}{(x-2)(x^2-5x+6)}\,dx$ (b) $\displaystyle\int \frac{8x^2+39x+11}{(x+2)(x^2-x-6)}\,dx$

(c) $\displaystyle\int \frac{8x^2-35x+3}{(x^2+1)(x-7)}\,dx$

Rj.-upute

(a) $$\frac{6x^2-19x+9}{(x-2)(x^2-5x+6)} = \ldots = \frac{5}{(x-2)^2} + \frac{6}{x-3}$$

$$\int \frac{6x^2-19x+9}{(x-2)(x^2-5x+6)}\,dx = \ldots = -\frac{5}{(x-2)} + 6\ln|x-3| + C$$

(b)

$$\frac{8x^2+39x+11}{(x+2)(x^2-x-6)} = \ldots = \frac{7}{(x+2)^2} + \frac{8}{x-3}$$

$$\int \frac{8x^2+39x+11}{(x+2)(x^2-x-6)}\,dx = \ldots = -\frac{7}{x+2} + 8\ln|x-3| + C$$

(c) $$\frac{8x^2-35x+3}{(x^2+1)(x-7)} = \ldots = \frac{5x}{x^2+1} + \frac{3}{x-7}$$

$$\int \frac{8x^2-35x+3}{(x^2+1)(x-7)}\,dx = \ldots = \frac{5}{2}\ln|x^2+1| + 3\ln|x-7| + C$$

(#) Odrediti integral $\displaystyle\int \frac{x^3-3}{x^4+10x^2+25}\,dx$.

Rj.-upute:

$$\frac{x^3-3}{x^4+10x^2+25} = \frac{x^3-3}{(x^2+5)^2} = \frac{Ax+B}{x^2+5} + \frac{Cx+D}{(x^2+5)^2} \qquad / \cdot (x^2+5)^2$$

$$x^3-3 = Ax(x^2+5) + B(x^2+5) + Cx + D$$

$x^3$: $A=1$

$x^2$: $B=0$

$x$: $5A + C = 0 \Rightarrow C = -5A$

$C = -5$

$x^0$: $5B + D = -3$

$\Rightarrow$

$D = -3-5B$

$D = -3$

$$\int \frac{x^3-3}{x^4+10x^2+25}\,dx = \int\left(\frac{x}{x^2+5} + \frac{-5x-3}{(x^2+5)^2}\right)dx =$$

$$= \left| \begin{array}{l} d(x^2+5) = 2x\,dx \\ x\,dx = \frac{1}{2}\,d(x^2+5) \\ -5x\,dx = -\frac{5}{2}\,d(x^2+5) \end{array} \right| = \frac{1}{2}\int \frac{d(x^2+5)}{x^2+5} - \frac{5}{2}\int \frac{d(x^2+5)}{(x^2+5)^2} - 3\int \frac{dx}{(x^2+5)^2}$$

$$= \left| \begin{array}{l} \text{za treći integral} \\ \text{uvodimo smjenu} \\ x = \sqrt{5}\,\mathrm{tg}\,z = \sqrt{5}\,\frac{\sin z}{\cos z} \\ dx = \frac{\sqrt{5}\,dz}{\cos^2 z} \end{array} \right| = \ldots = \frac{1}{2}\ln(x^2+5) + \frac{25-3x}{10(x^2+5)} - \frac{3}{10\sqrt{5}}\,\mathrm{arctg}\,\frac{x}{\sqrt{5}} + C$$

(#) Izračunati integrale

a) $\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} x|\cos x|\,dx$ b) $\int_{0}^{2\pi} x|\sin x|\,dx$ c) $\int_{0}^{2\pi} e^{x}|\sin x|\,dx$ d) $\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{x}|\cos x|\,dx$

Rj.

a)

$$|\cos x| = \begin{cases} -\cos x, & \cos x < 0 \\ \cos x, & \cos x \geq 0 \end{cases} = \begin{cases} \cos x, & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ -\cos x, & x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \end{cases}$$

$$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} x|\cos x|\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x\,dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x\cos x\,dx \overset{(*)}{=}$$

$$\int x\cos x\,dx = \begin{vmatrix} u=x & dv=\cos x\,dx \\ du=dx & v=\sin x \end{vmatrix} = x\sin x - \int \sin x\,dx = x\sin x + \cos x$$

$$\overset{(*)}{=} \left.\left(x\sin x + \cos x\right)\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overset{\frac{\pi}{2}-\frac{\pi}{2}}{} - \left.\left(x\sin x + \cos x\right)\right|_{\pi/2}^{\frac{3\pi}{2}} \overset{-\frac{3\pi}{2}-\frac{\pi}{2}}{} = 0 - (-2\pi) = 2\pi$$

b)

$$|\sin x| = \begin{cases} -\sin x, & \sin x < 0 \\ \sin x, & \sin x \geq 0 \end{cases} = \begin{cases} \sin x, & x \in [0, \pi] \\ -\sin x, & x \in (\pi, 2\pi) \end{cases}$$

$$\int x\sin x\,dx = \begin{vmatrix} u=x & dv=\sin x\,dx \\ du=dx & v=-\cos x \end{vmatrix} = -x\cos x + \int \cos x\,dx = \sin x - x\cos x$$

$$\int_{0}^{2\pi} x|\sin x|\,dx = \int_{0}^{\pi} x\sin x\,dx - \int_{\pi}^{2\pi} x\sin x\,dx = \left.\left(\sin x - x\cos x\right)\right|_{0}^{\pi} \overset{\pi-0}{} - \left.\left(\sin x - x\cos x\right)\right|_{\pi}^{2\pi} \overset{-2\pi-\pi}{} =$$

$$= 4\pi$$

c) $$\int_0^{2\pi} e^x |\sin x|\, dx = \int_0^{\pi} e^x \sin x\, dx - \int_{\pi}^{2\pi} e^x \sin x\, dx \overset{(*)}{=}$$

$$I = \int e^x \sin x\, dx = \begin{vmatrix} u = e^x & dv = \sin x\, dx \\ du = e^x dx & v = -\cos x \end{vmatrix} = -e^x \cos x +$$

$$+ \int e^x \cos x\, dx = \begin{vmatrix} u = e^x & dv = \cos x\, dx \\ du = e^x dx & v = \sin x \end{vmatrix} = -e^x \cos x + e^x \sin x - \underbrace{\int e^x \sin x dx}_{I}$$

$$\Rightarrow \quad 2I = e^x(\sin x - \cos x) \quad \Rightarrow \quad I = \frac{1}{2} e^x (\sin x - \cos x)$$

$$\overset{(*)}{=} \frac{1}{2} e^x (\sin x - \cos x)\Big|_0^{\pi} - \frac{1}{2} e^x (\sin x - \cos x)\Big|_{\pi}^{2\pi} = \frac{1}{2}\left(e^{2\pi} + 2e^{\pi} + 1\right)$$

$$= \frac{1}{2}\left(e^{\pi} + 1\right)^2$$

d) $$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} e^x |\cos x|\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \cos x\, dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^x \cos x\, dx =$$

$$= \frac{1}{2} e^x (\cos x + \sin x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{1}{2} e^x (\cos x + \sin x)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} =$$

$$= \frac{1}{2}\left(e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}\right) - \frac{1}{2}\left(-e^{\frac{3\pi}{2}} - e^{\frac{\pi}{2}}\right) = \frac{1}{2}\left(e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}} + e^{\frac{3\pi}{2}} + e^{\frac{\pi}{2}}\right)$$

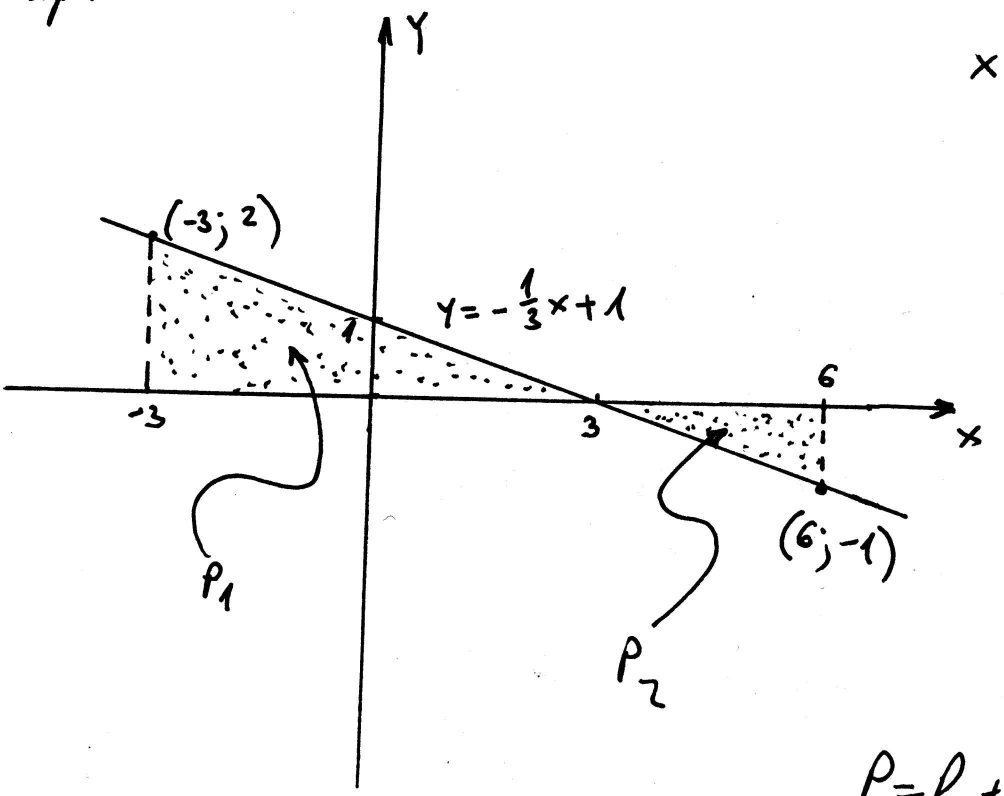
$$2e^{\frac{\pi}{2}} + e^{\frac{3\pi}{2}} + e^{-\frac{\pi}{2}}$$

$$= \frac{1}{2} e^{-\frac{\pi}{2}}\left(e^{\pi} + 2e^{\pi} + 1\right) =$$

$$= \frac{1}{2} e^{-\frac{\pi}{2}}\left(e^{\pi} + 1\right)^2$$

(#) Primenom određenog integrala odrediti površinu figure koju ograničava x-osa zajedno sa linijama $x+3y-3=0$, $x=-3$ i $x=6$.

Rj.-upute



$$x+3y-3=0$$

$$-3y=x-3 \quad /:(-3)$$

$$y=-\frac{1}{3}x+1$$

$$P=P_1+P_2$$

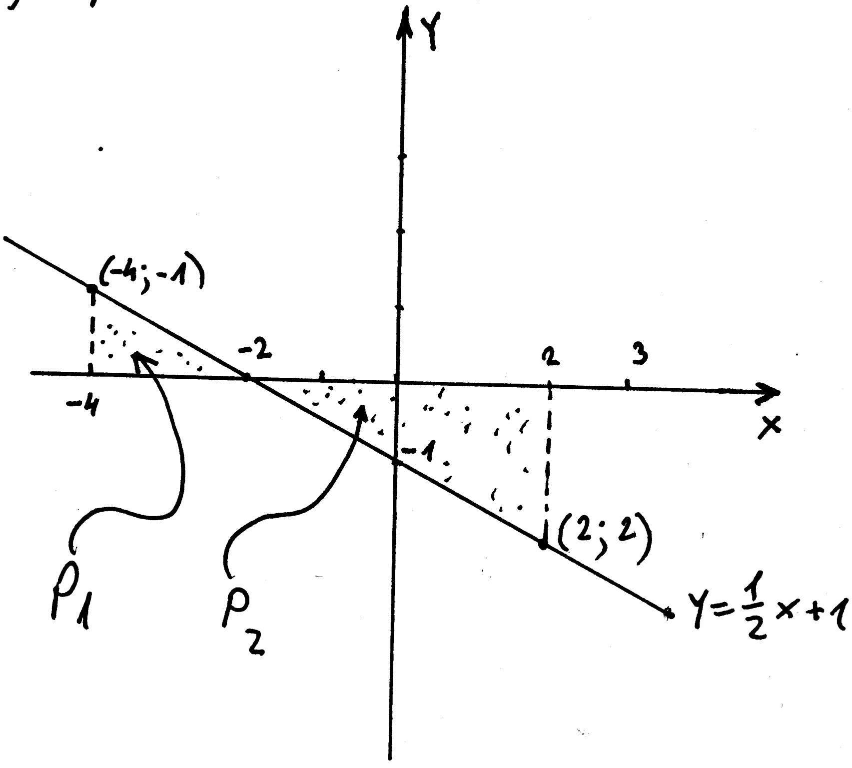
$$P_1=\int_{-3}^{3}\left(-\frac{1}{3}x+1\right)dx=\ldots=6$$

$$P_2=\left|\int_{3}^{6}\left(-\frac{1}{3}x+1\right)dx\right|=\ldots=+\frac{3}{2}$$

$$P=6+\frac{3}{2}=\frac{15}{2}$$

(#) Primenom određenog integrala Odrediti površinu figure koju ograničava x-osa zajedno sa linijama $-x-2y+2=0$, $x=-4$ i $x=2$.

Rj.-upute:



$$-x-2y+2=0$$

$$2y=-x+2$$

$$y=-\frac{1}{2}x+1$$

$$P=P_1+P_2$$

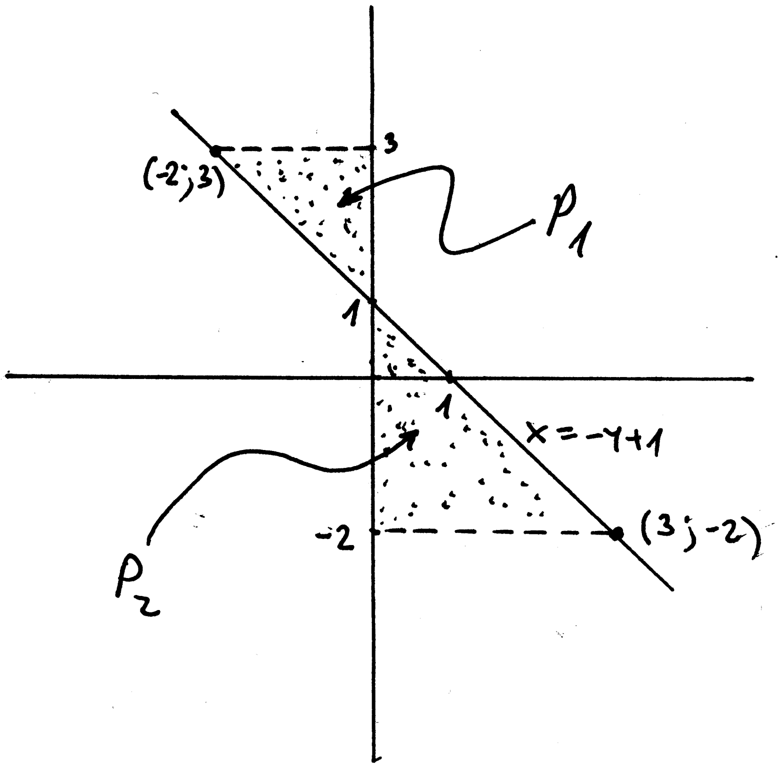
$$P_1=\int_{-4}^{-2}\left(-\frac{1}{2}x+1\right)dx=\ldots=1$$

$$P_2=\left|\int_{-2}^{2}\left(-\frac{1}{2}x+1\right)dx\right|=\ldots=4$$

$$P=P_1+P_2=5$$

(#) Primjerom određenog integrala odrediti površinu figure koju ograničava y-osa zajedno sa linijama $x+y-1=0$, $y=3$ i $y=-2$.

Rj.-upute



$$x+y-1=0$$

$$x=-y+1$$

$$P=P_1+P_2$$

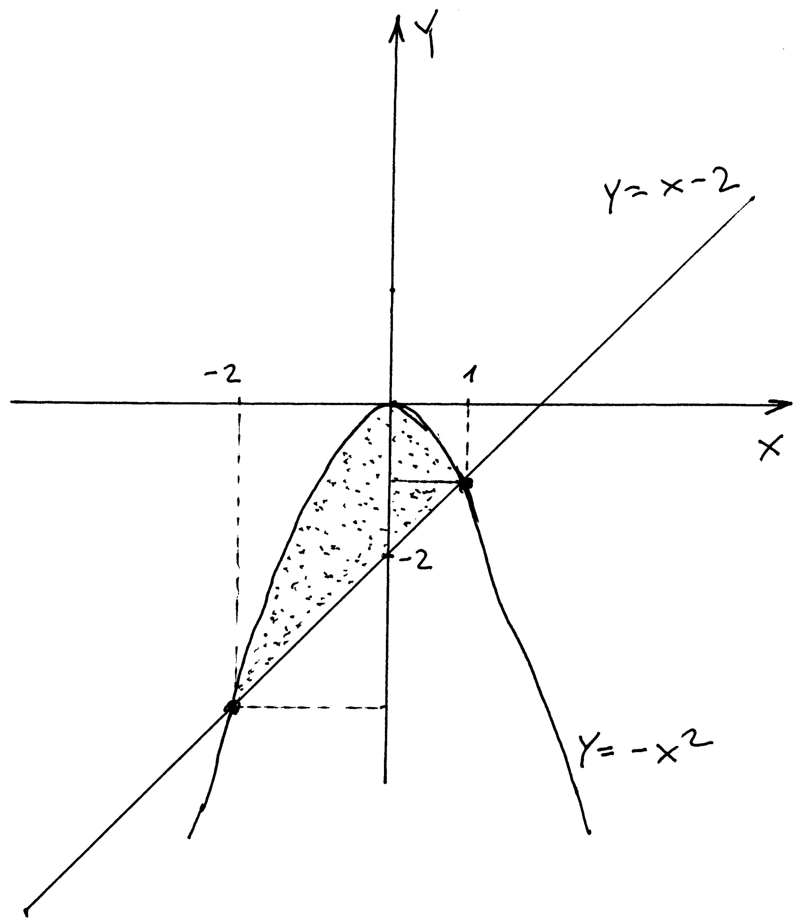
$$P_1=\left|\int_1^3(-y+1)\,dy\right|=\ldots=2$$

$$P_2=\int_{-2}^1(-y+1)\,dy=\ldots=\frac{9}{2}$$

$$P=2+\frac{9}{2}=\frac{13}{2}$$

#) Naći površinu figure ograničene linijama $y=-x^2$, $x-y-2=0$.

Rj. Nacrtajmo sliku



Pronađimo presječne tačke krive $y=-x^2$ i prave $x-y-2=0$.

$$\begin{array}{l} y=-x^2 \\ x-y-2=0 \\ \hline x+x^2-2=0 \end{array}$$

$$x^2+x-2=0$$

$$D=1+8=9 \quad x_{1,2}=\frac{-1\pm 3}{2}$$

$$x_1=-2, \quad x_2=1$$

$$(x-1)(x+2)=0$$

$$x=1 \Rightarrow y=-1$$

$$x=-2 \Rightarrow y=-4$$

I način:

$$P=\int_{-2}^{1}\left(-x^2-(x-2)\right)dx=\int_{-2}^{1}(-x^2-x+2)\,dx=\overbrace{-\frac{1}{3}x^3\Big|_{-2}^{1}}^{1+8}\overbrace{-\frac{1}{2}x^2\Big|_{-2}^{1}}^{1-4}\overbrace{+2x\Big|_{-2}^{1}}^{1+2}=$$

$$=-\frac{1}{3}\cdot 9+\frac{1}{2}\cdot 3+2\cdot 3=-3+\frac{3}{2}+6=-3-\frac{3}{2}=\frac{9}{2}$$

II način:

$$P=\iint_D dx\,dy \quad \text{gdje je} \quad D:\begin{cases} -2\le x\le 1 \\ x-2\le y\le -x^2 \end{cases}$$

$$P=\iint_D dx\,dy=\int_{-2}^{1}dx\int_{x-2}^{-x^2}dy=\int_{-2}^{1}\left((-x^2)-(x-2)\right)dx=\ldots=\frac{9}{2}$$

(#) Izračunati površinu ravne figure koja je ograničena parabolama $y=-x^2-4x$ i $y=x^2+2x$.

Rj.:

Za parabolu $y=-x^2-4x$ znamo da je $\cap$ oblika. Vidimo da x-osu siječe u tačkama $-4$ i $0$.

$y'=-2x-4$ Tjeme ove parabole je $T(-2,4)$

$-2x-4=0$

$x=-2$

Za parabolu $y=x^2+2x$ znamo da je $\cup$ oblika. Vidimo da x-osu siječe u tačkama $-2$ i $0$.

$y'=2x+2$ Tjeme ove parabole je $T(-1,1)$

$2x+2=0$

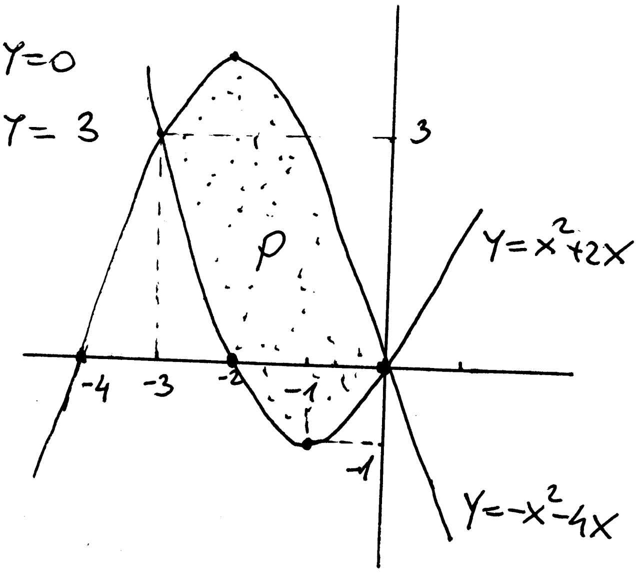
$x=-1$

Pronađimo još presječne tačke dvije date parabole.

$x=0 \Rightarrow y=0$

$x=-3 \Rightarrow y=3$

$$\begin{array}{l} y=-x^2-4x \\ y=x^2+2x \\ \hline -2x^2-6x=0 \quad /:(-2) \\ x^2+3x=0 \\ x(x+3)=0 \end{array}$$



$$P=\int_{-3}^{0}\left[(-x^2-4x)-(x^2+2x)\right]dx=\int_{-3}^{0}(-2x^2-6x)\,dx=-\frac{2}{3}x^3\Big|_{-3}^{0}-6\cdot\frac{1}{2}x^2\Big|_{-3}^{0}=$$

$$=-\frac{2}{3}(0-(-27))-3(0-9)=-18+27=9$$ vrijednost tražene površine

(#) Izračunati površinu ravne figure koja je ograničena parabolama $Y=4-x^2$ i $Y=x^2-2x$.

Rj.

$Y = 4-x^2 = -x^2+4$

za ovu parabolu znamo da je oblika ∩
vidimo da x-osu sijeće u tačkama 2 i -2

$Y = x^2-2x$

je parabola ∪ oblika
x-osu siječe u tačkama 0 i 2.

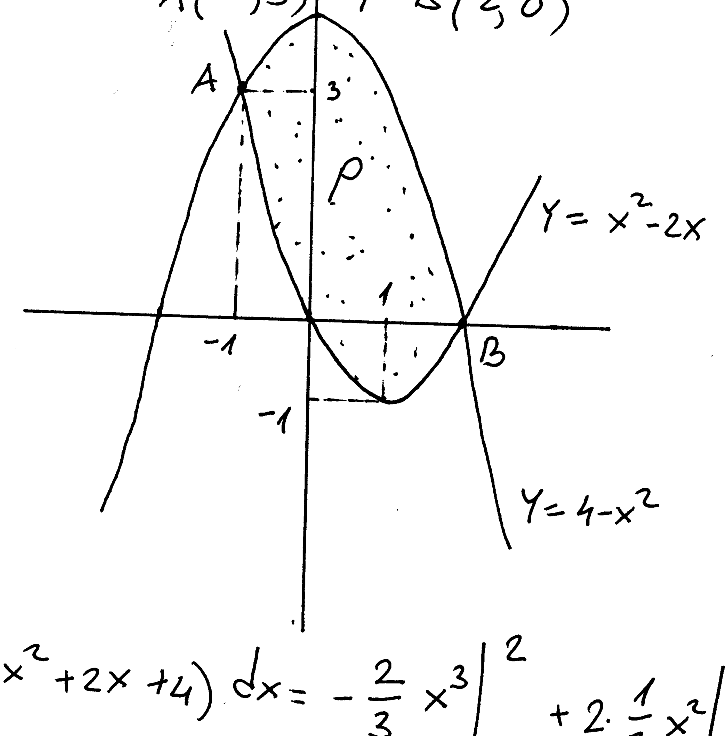
Pronađimo još presječne tačke dvije date parabole

$$\begin{array}{rl} & Y=-x^2+4 \\ - & Y=x^2-2x \\ \hline & -2x^2+2x+4=0 \quad /:(-2) \\ & x^2-x-2=0 \\ & (x-2)(x+1)=0 \end{array}$$

$x=-1 \Rightarrow Y=3$

$x=2 \Rightarrow Y=0$

Presječne tačke parabola su $A(-1,3)$ i $B(2,0)$



$$P=\int_{-1}^{2}\left[(4-x^2)-(x^2-2x)\right]dx=\int_{-1}^{2}(-2x^2+2x+4)\,dx=-\frac{2}{3}x^3\Big|_{-1}^{2}+2\cdot\frac{1}{2}x^2\Big|_{-1}^{2}$$

$$+4x\Big|_{-1}^{2}=-\frac{2}{3}(8+1)+(4-1)+4(2+1)=-6+3+12=9$$

tražena površina

(#) Izračunati površinu ravne figure koja je ograničena krivim linijama $x=y^2-1$ i $x=-y^2-2y+3$

Rj: Za krivu $x=y^2-1$ vidimo da je sljedećeg oblika ⊂ y-osu sječe u tačkama -1 i 1

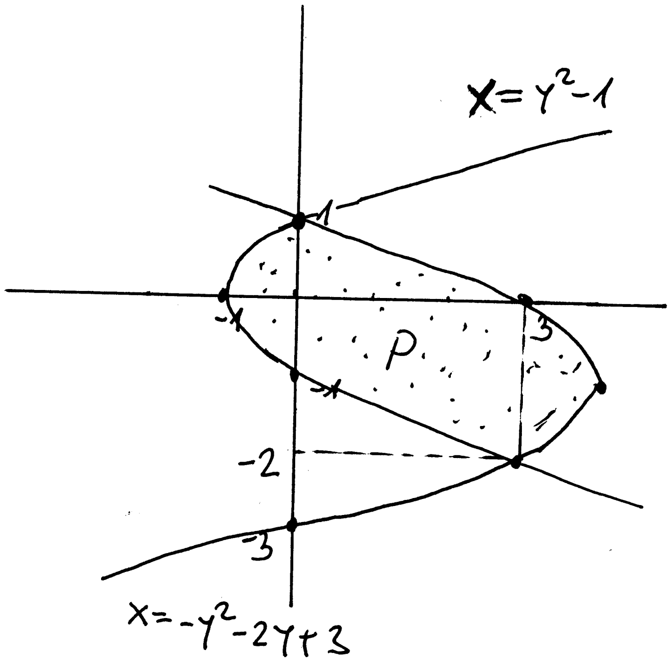
Kriva $x=-y^2-2y+3$ je oblika ⊃. y-osu sječe u tačkama -3 i 1.

Pronađimo presječne tačke dvije date krive.

$$\begin{array}{l} x = y^2-1 \\ -\quad x=-y^2-2y+3 \\ \hline 2y^2+2y-4=0 \quad /:2 \\ y^2+y-2=0 \\ (y+2)(y-1)=0 \end{array}$$

$y=-2 \Rightarrow x=3$

$y=1 \Rightarrow x=0$



$$P=\int_{-2}^{1}\left[(-y^2-2y+3)-(y^2-1)\right]dy=$$

$$=\int_{-2}^{1}(-2y^2-2y+4)\,dy=-\frac{2}{3}y^3\Big|_{-2}^{1}-y^2\Big|_{-2}^{1}+4y\Big|_{-2}^{1}=$$

24·3= 7

$$=-\frac{2}{3}(1-(-8))-(1-4)+4(1-(-2))=$$

$$=-6+3+12=9$$ tražena površina

(#) Izračunati površinu ravne figure koja je ograničena parabolama $x=y^2-4y+3$ i $x=-y^2+2y+3$.

Rj:

Parabola $x=y^2-4y+3$ je C oblika. $x=(y-3)(y-1)$

y-osu siječe u tačkama 1 i 3.

$x'=2y-4$

$2y-4=0$ $\quad x=4-8+3=-1$ Tjeme ove parabole je $T(-1, 2)$

$y=2$

Parabola $x=-y^2+2y+3$ je Ɔ oblika. $x=-(y+1)(y-3)$

y-osu siječe u tačkama $-1$ i 3.

$x'=-2y+2$

$-2y+2=0$ $\quad x=-1+2+3=4$ Tjeme ove parabole je $T(4, 1)$

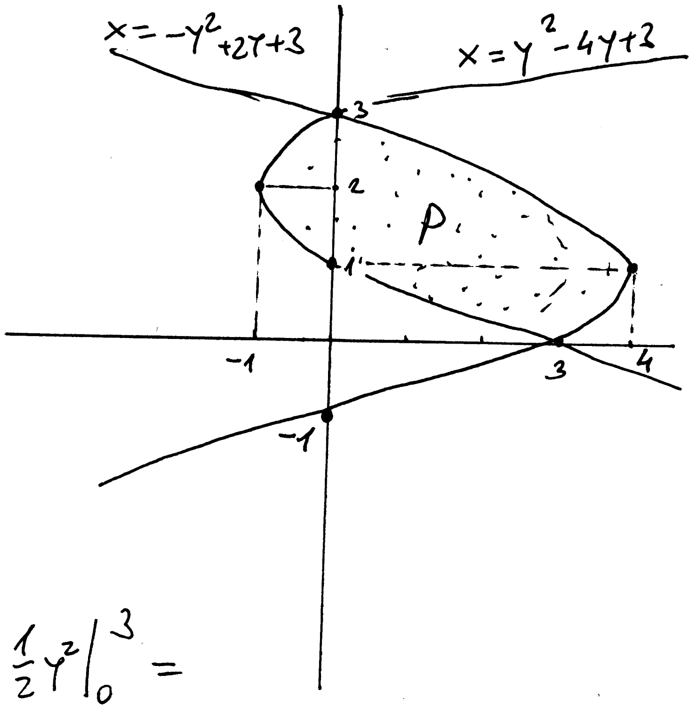
$y=1$

Pronađimo još presječne tačke dvije date parabole

$$\begin{array}{rl} & x=y^2-4y+3 \\ - & x=-y^2+2y+3 \\ \hline & 2y^2-6y=0 \\ & 2y(y-3)=0 \end{array}$$

$y=0 \Rightarrow x=3$

$y=3 \Rightarrow x=0$



$$P=\int_0^3\left[(-y^2+2y+3)-(y^2-4y+3)\right]dy=$$

$$=\int_0^3(-2y^2+6y)\,dy=-\frac{2}{3}y^3\Big|_0^3+6\cdot\frac{1}{2}y^2\Big|_0^3=$$

$$=-2\cdot 9+3\cdot 9=9$$ tražena površina

(#) Odrediti površinu figure ograničene hiperbolom $xy=4$ i pravom $y=-x-5$.

Rj.: -upute

Odredimo presječne tačke datih krivih i skicirajmo sliku.

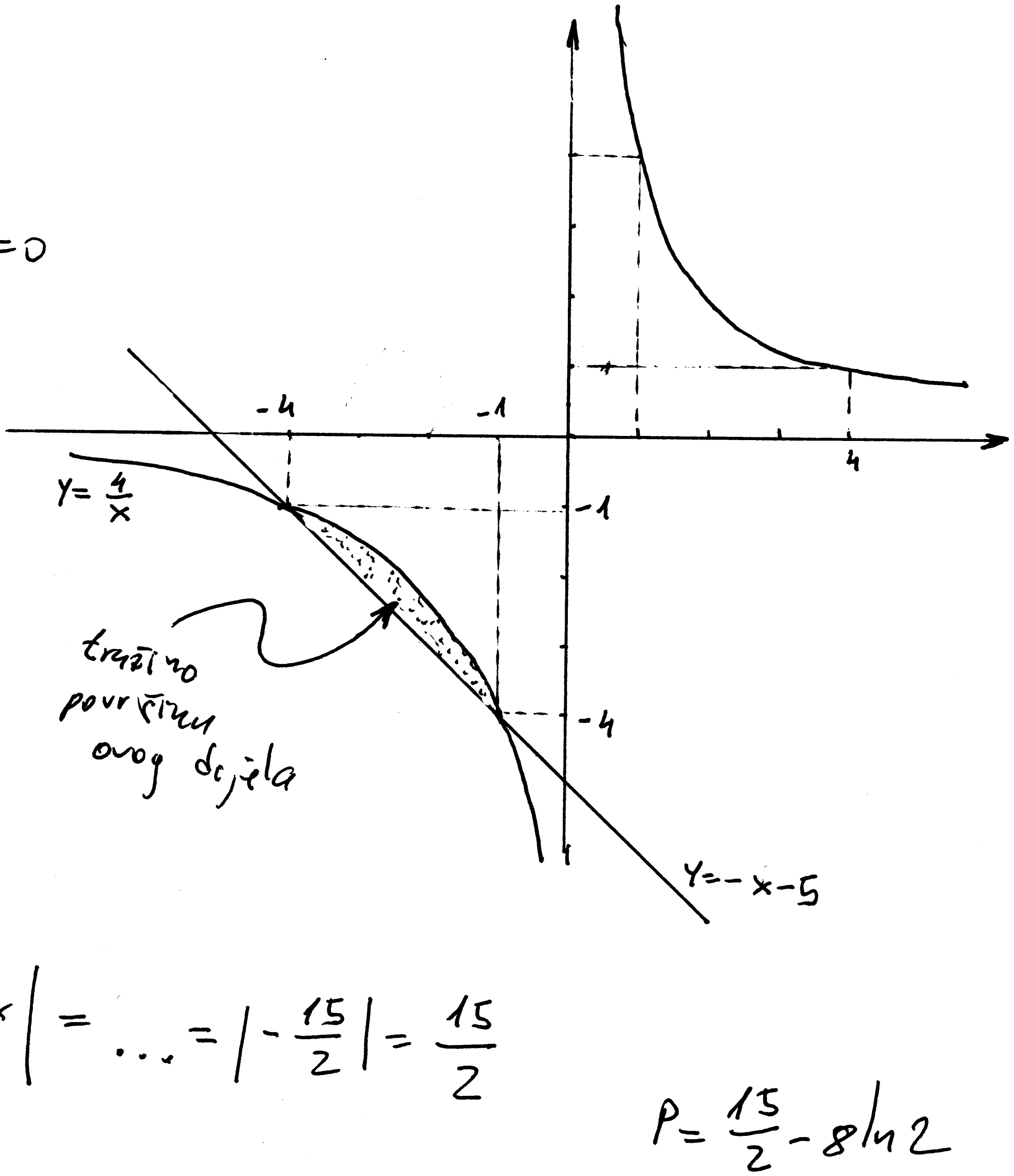
$$\begin{array}{l} xy=4 \\ y=-x-5 \\ \hline x(-x-5)=4 \end{array}$$

$$x^2+5x+4=0$$

$$(x+4)(x+1)=0$$

$x_1=-4 \Rightarrow y_1=-1$

$x_2=-1 \Rightarrow y_2=-4$



tražimo površinu ovog dijela

$P=P_1-P_2$

$$P_1=\left|\int_{-4}^{-1}(-x-5)dx\right| = \ldots = \left|-\frac{15}{2}\right| = \frac{15}{2}$$

$$P_2=\left|\int_{-4}^{-1}\frac{4}{x}\,dx\right| = \ldots = |-8\ln 2| = 8\ln 2$$

$P=\frac{15}{2}-8\ln 2$ tražena površina

(#) Odrediti površinu figure ograničenu parabolom $y=x^2+4x$ i pravom $x-y+4=0$.

Rj.-upute:

Odredimo presječne tačke parabole i prave

$$y=x^2+4x$$
$$y=x+4$$
$$x^2+4x=x+4$$
$$x^2+3x-4=0$$
$$(x+4)(x-1)=0$$

$x_1=-4 \Rightarrow y_1=0$

$x_2=1 \Rightarrow y_2=5$

$y=x^2+4x=x(x+4)$

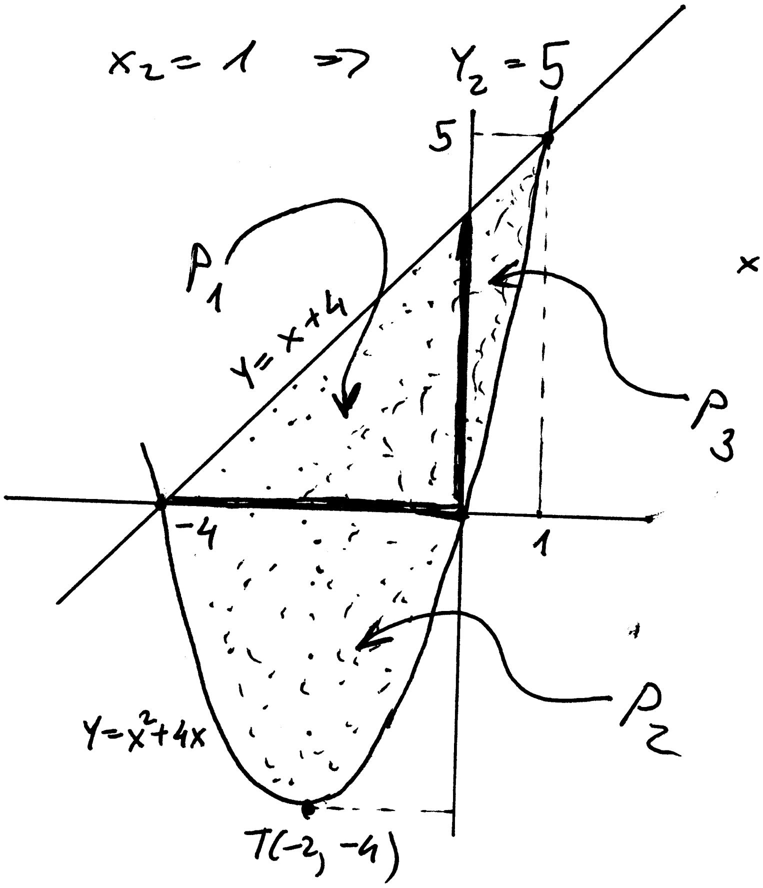
$y'=2x+4$

$2x+4=0$

$x=-2$

$x=-2 \Rightarrow y=-4$

$T(-2;-4)$



$$P=P_1+P_2+P_3$$

$$P_1=\int_{-4}^{0}(x+4)\,dx=\ldots=8$$

$$P_2=\left|\int_{-4}^{0}(x^2+4x)\,dx\right|=\ldots=\left|-\frac{32}{3}\right|=\frac{32}{3}$$

$$P_3=\int_{0}^{1}\left((x+4)-(x^2+4x)\right)dx=\ldots=\frac{13}{6}$$

$$P=P_1+P_2+P_3=8+\frac{32}{3}+\frac{13}{6}=\frac{125}{6}$$ tražena površina

(#) Odrediti površinu figure ograničenu parabolom $4y = 8x - x^2$ i pravom $4y = x + 6$.

Rj.-upute:

Prvo odredimo presječne tačke parabole i prave

$$\begin{aligned} 4y &= 8x - x^2 \\ 4y &= x + 6 \\ \hline 8x - x^2 &= x + 6 \end{aligned}$$

$$x^2 - 7x + 6 = 0$$

$$D = 49 - 24 = 25$$

$$x_{1,2} = \frac{7 \pm 5}{2} \qquad x_1 = 1 \quad x_2 = 6$$

$x_1 = 1 \Rightarrow 4y = 7$, $y_1 = \frac{7}{4}$

$x_2 = 6 \Rightarrow 4y = 12$, $y = 3$

Presječne tačke prave i parabole su $A(1; \frac{7}{4})$ i $B(6; 3)$

$$y = -\frac{1}{4}x^2 + 2x = x(-\frac{1}{4}x + 2)$$

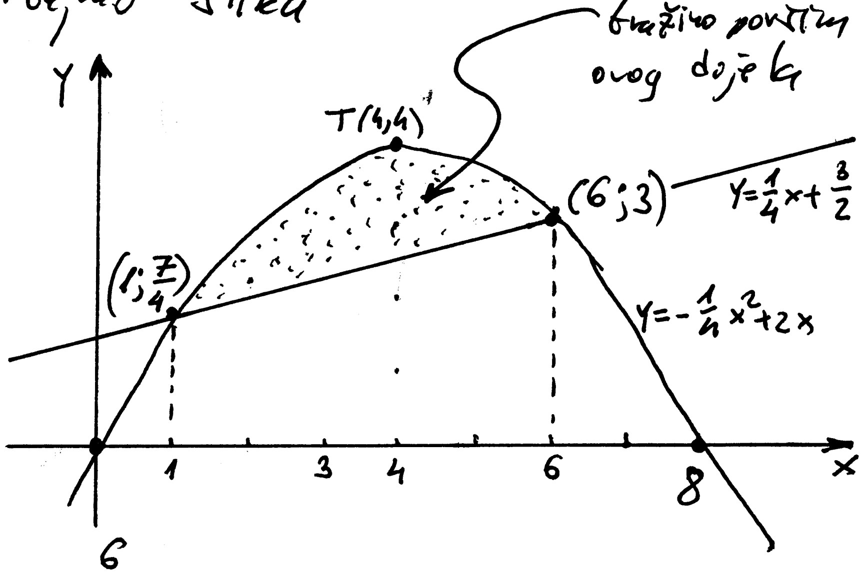
$$y' = -\frac{1}{2}x + 2$$

$$-\frac{1}{2}x + 2 = 0 \quad / \cdot 2$$

$$x = 4$$

$$T(4, 4)$$

Nacrtajmo sliku



$$P = P_1 - P_2 \quad \text{gdje je} \quad P_1 = \int_1^6 (-\frac{1}{4}x^2 + 2x)\,dx = \ldots = \frac{205}{12}$$

$$P_2 = \int_1^6 (\frac{1}{4}x + \frac{3}{2})\,dx = \ldots = \frac{95}{8}$$

$$P = P_1 - P_2 = \frac{205}{12} - \frac{95}{8} = \frac{410 - 285}{24} = \frac{125}{24} = 5\frac{5}{24}$$

(#) Odrediti površinu figure ograničene hiperbolom $xy=6$ i pravom $y=7-x$.

Rj.-upute:
Prvo odredimo presječne tačke date hiperbole i prave

$$\frac{\begin{array}{l} xy=6 \\ y=7-x \end{array}}{}$$

$x(7-x)=6$

$7x-x^2=6$

$x^2-7x+6=0$

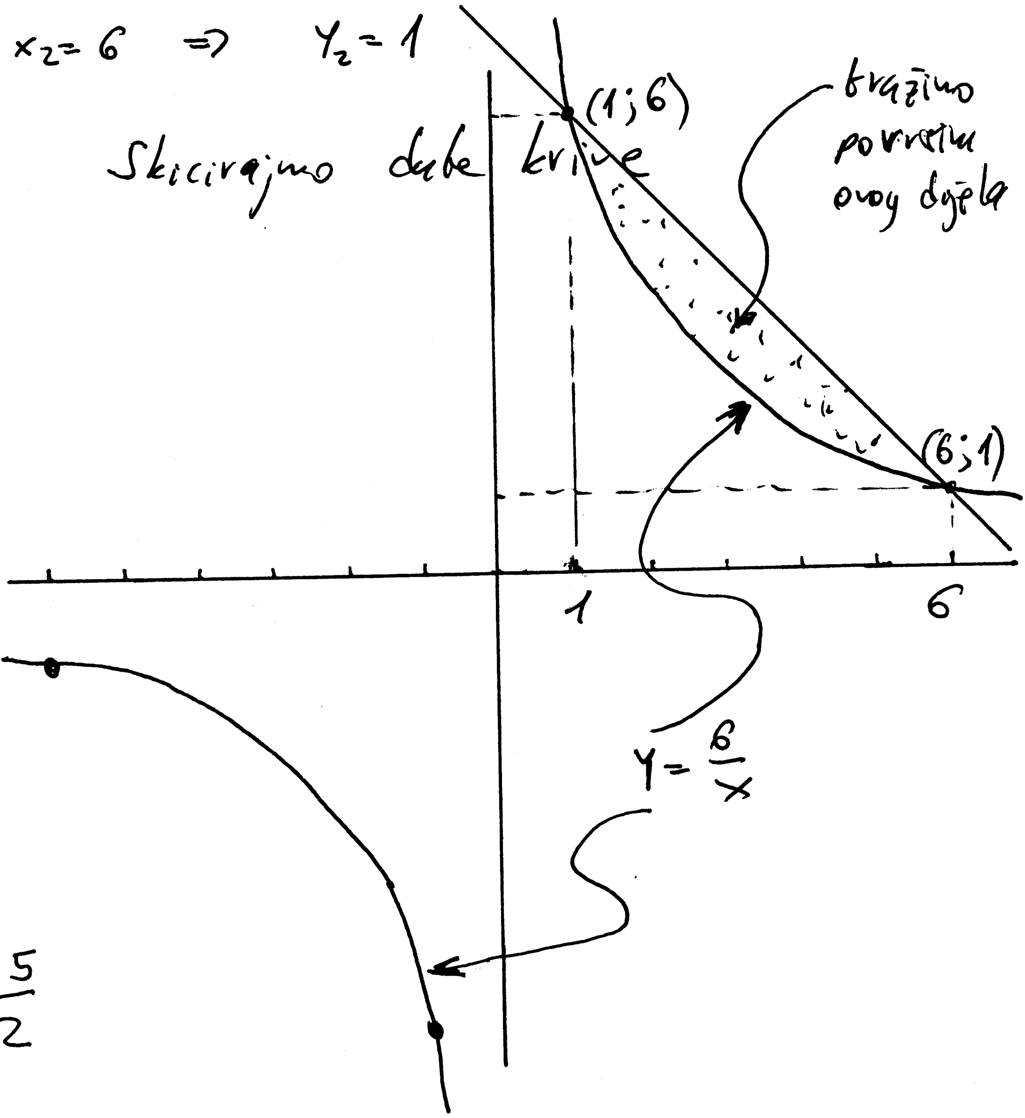
$D=49-24=25$

$x_{1,2}=\frac{7\pm 5}{2}$

$x_1=1 \Rightarrow y_1=6$

$x_2=6 \Rightarrow y_2=1$

Skicirajmo date krive



$P=P_1-P_2$

$P_1=\int_1^6 (7-x)\,dx= \ldots = \frac{35}{2}$

$P_2=\int_1^6 \frac{6}{x}\,dx= \ldots = 6\ln 6$

$P=\frac{35}{2}-6\ln 6$

(#) Odrediti površinu figure ograničene parabolom $4x = 8y - y^2$ i pravom $4x = y + 6$.

Rj.-upute

$$\begin{aligned} 4x &= 8y - y^2 \\ 4x &= y + 6 \end{aligned}$$

---

$8y - y^2 = y + 6$

$y^2 - 7y + 6 = 0$

$(y-1)(y-6) = 0$

$y_1 = 1$

$y_2 = 6$

$y = 1 \Rightarrow 4x = 1+6$

$4x = 7$

$x = \frac{7}{4}$

$y = 6 \Rightarrow 4x = 6+6$

$x = 3$

Parabola i prava se sijeku u tačkama $(\frac{7}{4}; 1)$ i $(3; 6)$

$4x = 8y - y^2 \quad /:4$

$x = 2y - \frac{1}{4}y^2$

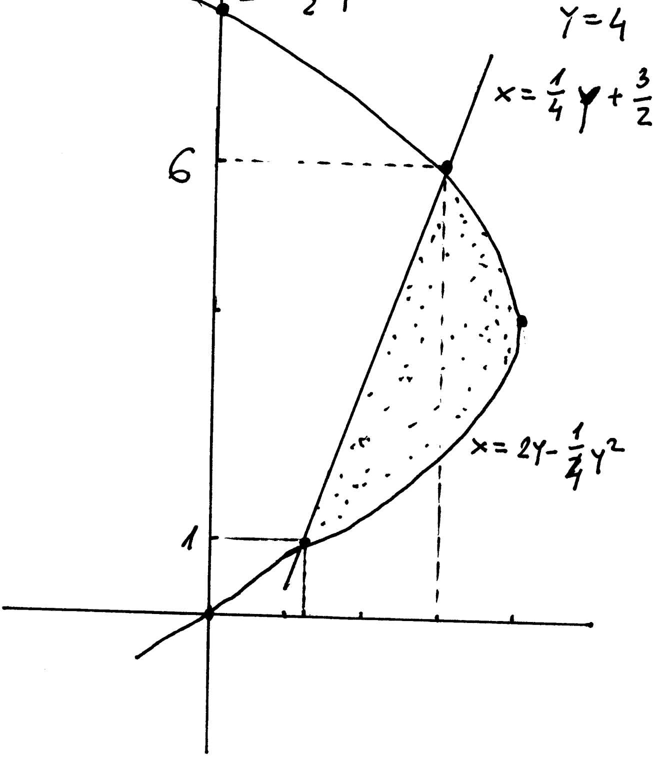
$x' = 2 - \frac{1}{2}y$

$2 - \frac{1}{2}y = 0$

$\frac{1}{2}y = 2$

$y = 4$ $\Rightarrow$ Tjeme parabole $4x = 8y - y^2$ je u tački $T(4; 4)$

$4x = 32 - 16$



$$P = \int_1^6 \left[ \left(2y - \frac{1}{4}y^2\right) - \left(\frac{1}{4}y + \frac{3}{2}\right) \right] dy =$$

$$= \int_1^6 \left(-\frac{1}{4}y^2 + \frac{7}{4}y - \frac{3}{2}\right) dy = \dots =$$

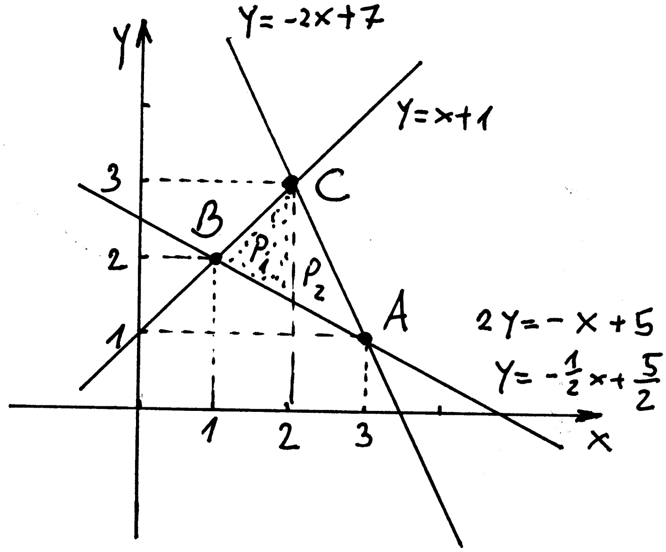
$= \frac{125}{24}$ tražena površina

(#) Primjenom određenog integrala izračunati površinu figure koju ograničavaju linije $x+2y-5=0$, $2x+y-7=0$ i $y=x+1$.

Rj. -upute:

Date su tri prave. Odredimo presječne tačke pravih i nacrtajmo sliku.

$$\begin{array}{c} x+2y-5=0 \\ 2x+y-7=0 \\ \hline \vdots \\ A(3;1) \end{array} \qquad \begin{array}{c} x+2y-5=0 \\ y=x+1 \\ \hline \vdots \\ B(1;2) \end{array} \qquad \begin{array}{c} 2x+y-7=0 \\ y=x+1 \\ \hline \vdots \\ C(2;3) \end{array}$$



$$P=\int_1^2\left[(x+1)-\left(-\frac{1}{2}x+\frac{5}{2}\right)\right]dx+$$

$$+\int_2^3\left[(-2x+7)-\left(-\frac{1}{2}x+\frac{5}{2}\right)\right]dx$$

$$P_1=\int_1^2\left(\frac{3}{2}x-\frac{3}{2}\right)dx=\ldots=\frac{3}{4}$$

$$P_2=\int_2^3\left(-\frac{3}{2}x+\frac{9}{2}\right)dx=\ldots=\frac{3}{4}$$

$$P=\frac{3}{4}+\frac{3}{4}=\frac{3}{2}$$

tražena površina

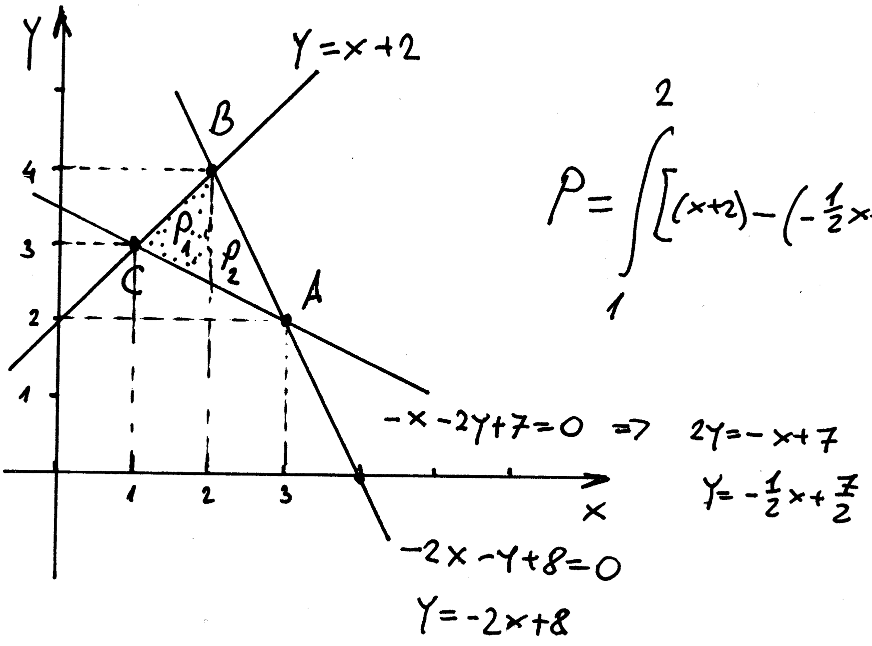
(#) Primjenom određenog integrala izračunati površinu figure koju ograničavaju linije $-2x-y+8=0$, $-x-2y+7=0$ i $y=x+2$.

Rj.-upute:

Date su tri prave. Odredimo presječne tačke pravih i nacrtajmo sliku.

$$\frac{\begin{array}{c}-2x-y+8=0\\ -x-2y+7=0\end{array}}{\vdots}\qquad \frac{\begin{array}{c}-2x-y+8=0\\ y=x+2\end{array}}{\vdots}\qquad \frac{\begin{array}{c}-x-2y+7=0\\ y=x+2\end{array}}{\vdots}$$

$$A(3;2)\qquad B(2;4)\qquad C(1;3)$$



$$P=\int_1^2\left[(x+2)-\left(-\frac{1}{2}x+\frac{7}{2}\right)\right]dx+\int_2^3\left[(-2x+8)-\left(-\frac{1}{2}x+\frac{7}{2}\right)\right]dx$$

$$P_1=\int_1^2\left(\frac{3}{2}x-\frac{3}{2}\right)dx=\ldots=\frac{3}{4}$$

$$P_2=\int_2^3\left(-\frac{3}{2}x+\frac{9}{2}\right)dx=\ldots=\frac{3}{4}$$

$$P=2\cdot\frac{3}{4}=\frac{3}{2}$$

tražena površina

(#) Primjenom određenog integrala izračunati površinu figure koju ograničavaju linije $y+2x+7=0$, $x+2y+5=0$ i $y=x-1$.

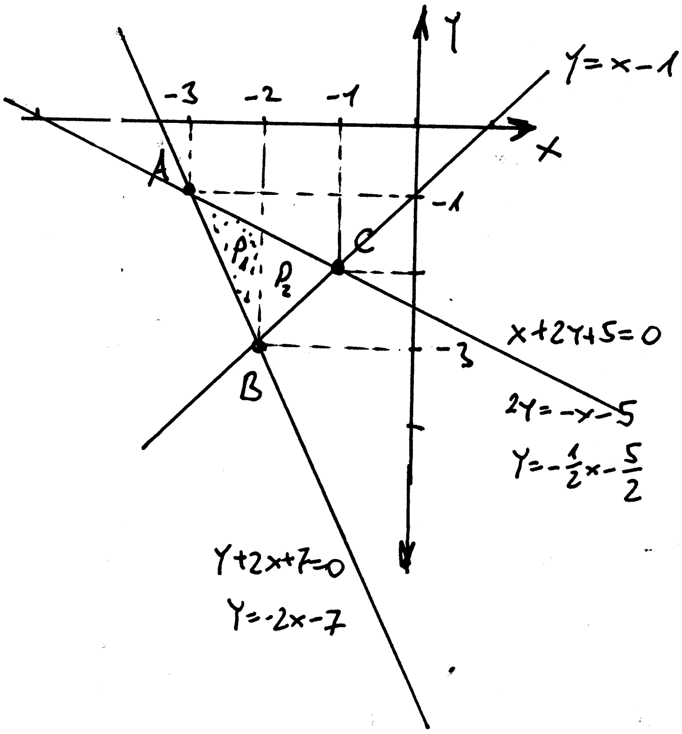
Rj.-upute:

Date su tri prave. Odredimo presječne tačke pravih i nacrtajmo sliku.

$$\frac{\begin{matrix} y+2x+7=0 \\ x+2y+5=0 \end{matrix}}{\vdots} \qquad A(-3;-1)$$

$$\frac{\begin{matrix} y+2x+7=0 \\ y=x-1 \end{matrix}}{\vdots} \qquad B(-2;-3)$$

$$\frac{\begin{matrix} x+2y+5=0 \\ y=x-1 \end{matrix}}{\vdots} \qquad C(-1;-2)$$



$$P=\left|\int_{-3}^{-2}\left[(-2x-7)-\left(-\frac{1}{2}x-\frac{5}{2}\right)\right]dx\right|+$$

$$+\left|\int_{-2}^{-1}\left[(x-1)-\left(-\frac{1}{2}x-\frac{5}{2}\right)\right]dx\right|$$

$$P_1=\left|\int_{-3}^{-2}\left(-\frac{3}{2}x-\frac{9}{2}\right)dx\right|=\ldots=\left|-\frac{3}{4}\right|=\frac{3}{4}$$

$$P_2=\left|\int_{-2}^{-1}\left(\frac{3}{2}x+\frac{3}{2}\right)dx\right|=\ldots=\left|-\frac{3}{4}\right|=\frac{3}{4}$$

$$\Rightarrow P=\frac{3}{2}$$

tražena površina

(#) Izračunati površinu ravne figure ograničene parabolom $Y=ax^2+bx$ koja sadrži tačke $A(-3;-3)$ i $B(-1;-3)$ i pravom $x=y-4$.

Rj. Odredimo prvo brojeve a i b iz jednačine parabole

$A(-3;-3) \Rightarrow \quad -3=9a-3b \quad /:(-3)$

$B(-1;-3) \Rightarrow \quad -3=a-b$

$-3a+b=1$

$+\quad a-b=-3$

$-2a=-2 \qquad b=a+3$

$a=1 \qquad b=4$

Data parabola ima jednačinu

$Y=x^2+4x$

Odredimo presječne tačke date parabole i prave

$Y=x+4$

$Y=x^2+4x$

$x^2+4x=x+4$

$x^2+3x-4=0$

$(x-1)(x+4)=0$

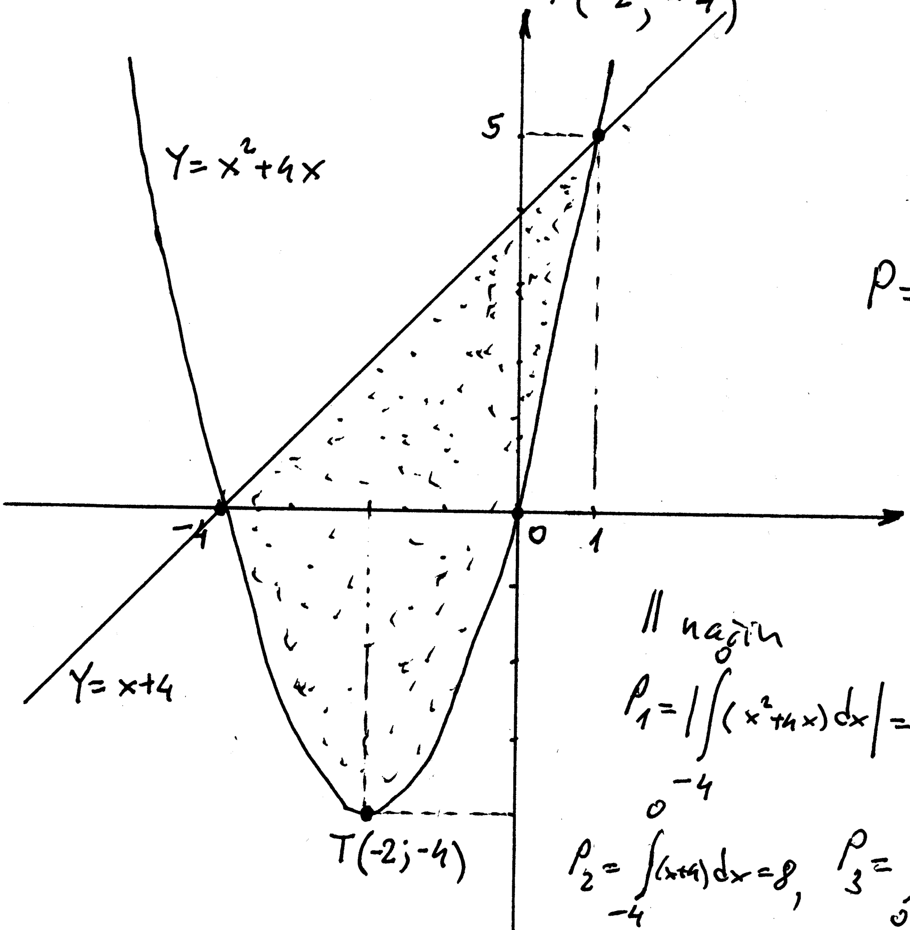
$x_1=1 \Rightarrow Y_1=5$

$x_2=-4 \Rightarrow Y_2=0$

$Y=x^2+4x=x(x+4)$

$Y'=2x+4$

$T(-2;-4)$



$$P=\int_{-4}^{1}\left[(x+4)-(x^2+4x)\right]dx=$$

$$=\int_{-4}^{1}(-x^2-3x+4)\,dx=$$

$$=\ldots=\frac{125}{6}$$

tražena površina

II način

$P_1=\left|\int_{-4}^{0}(x^2+4x)\,dx\right|=\frac{32}{3}$

$P_2=\int_{-4}^{0}(x+4)\,dx=8, \quad P_3=\int_{0}^{1}\left[(x+4)-(x^2+4x)\right]dx=\frac{13}{6}$

(#) Proverititi da li je data f-ja rješenje date diferencijalne jednačine

a) $y=\sqrt{x}$, $2yy'=1$

b) $\ln x \ln y = c$, $y\ln y\,dx + x\ln x\,dy = 0$

c) $s=-t-\frac{1}{2}\sin 2t$, $\frac{d^2s}{dt^2}+tg\,t\,\frac{ds}{dt}=\sin 2t$.

Rj.: a) $y=\sqrt{x}=(x)^{\frac{1}{2}}$

$$y'=\frac{1}{2\sqrt{x}} \qquad y\cdot y'=\sqrt{x}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{2}$$

$$2yy'=1$$

F-ja $y=\sqrt{x}$ je rješenje diferencijalne jednačine $2yy'=1$.

b) $\ln x \ln y = c \quad /d$

$$\frac{\partial(\ln x\ln y)}{\partial x}dx+\frac{\partial(\ln x\ln y)}{\partial y}dy=0$$

$$\ln y\cdot\frac{1}{x}dx+\ln x\cdot\frac{1}{y}dy=0$$

$$\ln x\,\frac{1}{y}dy=-\ln y\cdot\frac{1}{x}dx \quad /\cdot\frac{y}{\ln x}$$

$$dy=-\frac{y\ln y}{x\ln x}dx$$

Uvrstimo dobijeni izraz za dy u jednačinu $y\ln y\,dx+x\ln x\,dy=0$

$$y \ln y \, dx + x \ln x \left( - \frac{y \ln y}{x \ln x} dx \right) = 0$$

$$0 = 0$$

Prema tome f-ja $\ln x \ln y = c$ je rješenje diferencijalne jednačine $y \ln y \, dx + x \ln x \, dy = 0$.

c) $s = -t - \frac{1}{2} \sin 2t$

$$\frac{ds}{dt} = -1 - \frac{1}{2} \cdot 2 \cos 2t = -1 - \cos 2t$$

$$\frac{d^2 s}{dt^2} = 2 \sin 2t$$

$$\frac{d^2 s}{dt^2} + tg\, t \frac{ds}{dt} = \sin 2t$$

$$2 \sin 2t + tg\, t (-1 - \cos 2t) = \sin 2t$$

$$\sin 2t + tg\, t (\underline{-\sin^2 t} - \cos^2 t - \cos^2 t \underline{+ \sin^2 t}) = 0$$

$$\sin 2t - 2 \cos^2 t \, tg\, t = 0$$

$$\sin 2t - 2 \cos^2 t \frac{\sin t}{\cos t} = 0$$

$$\sin 2t - 2 \sin t \cos t = 0$$

$$0 = 0$$

F-ja $s = -t - \frac{1}{2} \sin 2t$ je rješenje diferencijalne jednačine $\frac{d^2 s}{dt^2} + tg\, t \frac{ds}{dt} = \sin 2t$.

(#) Ako znamo opšte rješenje $4x^2+y^2=C^2$ neke diferencijalne jednačine prvog reda, odrediti i grafički prikazati integralne krive (parcijalni integrali) koje prolaze kroz tačke $B_1(-1;0)$, $B_2(0;-3)$ i $B_3(2;0)$.

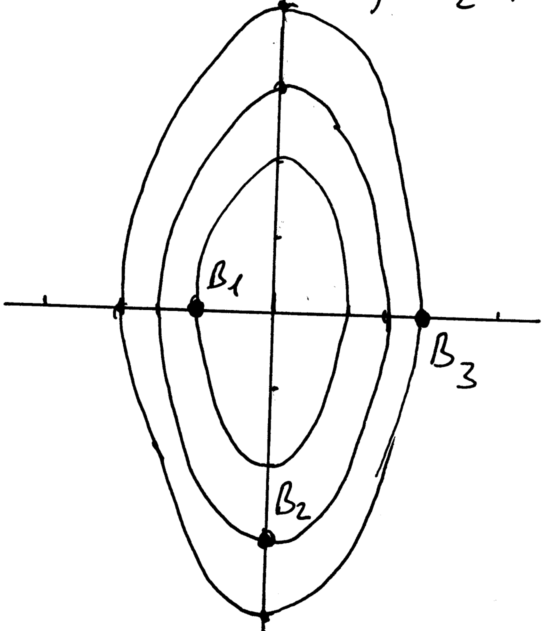
Rj: Opšte rješenje $F(x,y,C)=0$ diferencijalne jednačine prvog reda $f(x,y,y')=0$ geometrički definišu familiju krivih koje zavise samo od parametra C. Zamjenjujući u opšte rješenje koordinate tačke P, odredit ćemo vrijednost C, u kojoj opšte rješenje integralne krive, prolazi kroz tačku P.

Za tačku $B_1$: $4=C^2$; $4x^2+y^2=4$.

Za tačku $B_2$: $9=C^2$; $4x^2+y^2=9$.

Za tačku $B_3$: $16=C^2$; $4x^2+y^2=16$.

Za dobijene jednakosti integralne krive prolaze kroz tačke $B_1$, $B_2$ i $B_3$. Nacrtajmo ove krive.



Date krive su koncentrične elipse čiji je centar u koordinatnom početku

① Odrediti tip diferencijalne jednačine:

a) $yy' + x e^y = 0$

Rj. $yy' = -x e^y$

$y' = -x \frac{1}{y} e^y$ diferencijalna jednačina sa razdvojenim promenj.

b) $y + xy' = 4\sqrt{y'}$

Rj. $y = xy' + 4\sqrt{y'}$ Klerova difer. jedn.

c) $y' - y\,tg\,x + 2\sin x - 1 = 0$

Rj. $y' - tg\,x \cdot y = 1 - 2\sin x$ linearna difer. jedn.

d)

$xy' - y = (x+y) \ln \frac{x+y}{x}$

Rj. $xy' = y + (x+y)\ln(1+\frac{y}{x}) \quad /:x$

$y' = \frac{y}{x} + (1+\frac{y}{x})\ln(1+\frac{y}{x})$ homogena difer. jednačina

e) $xy' = y - xy\sin x$

Rj. $xy' = y(1 - x\sin x)$

$y' = y \cdot \frac{1 - x\sin x}{x}$ difer. jedn. sa razdvojenim promenljivim

f) $(x^2+1)y' - xy^2 = xy(x^2y - 1)$

Rj. $(x^2+1)y' - xy^2 = xy \cdot x^2 y - xy$

$(x^2+1)y' + xy = x^2 \cdot xy^2 + xy^2$

$(x^2+1)y' + xy = xy^2(x^2+1) \quad /:(x^2+1)$

$y' + \frac{x}{x^2+1} y = xy^2$

Bernulijeva diferenc. jedn.

(#) Odrediti opšte rješenje sljedećih diferencijalnih jednačina:

a) $(x+1)^3 dy - (y-2)^2 dx = 0$. b) $\frac{1}{\cos^2 x \cos y} dx = -ctg x \sin y \, dy$.

c) $(\sqrt{xy} + \sqrt{x}) y' - y = 0$. d) $2^{x+y} + 3^{x-2y} y' = 0$.

a)

$$(x+1)^3 dy - (y-2)^2 dx = 0 \quad /:(x+1)^3 (y-2)^2$$

$$\frac{dy}{(y-2)^2} - \frac{dx}{(x+1)^3} = 0 \quad /\int$$

iz oblika $\frac{dy}{(y-2)^2} = \frac{dx}{(x+1)^3}$ vidimo da je ovo diferencijalna jednačina sa razdvojenim promjenjivim

$$\int \frac{\overset{d(y-2)}{\overbrace{dy}}}{(y-2)^2} - \int \frac{\overset{= d(x+1)}{\overbrace{dx}}}{(x+1)^3} = c$$

$$\int (y-2)^{-2} d(y-2) - \int (x+1)^{-3} d(x+1) = c$$

$$-\frac{1}{y-2} + \frac{1}{2(x+1)^2} = c$$

opšte rješenje diferencijalne jednačine

b)

$$\frac{dx}{\cos^2 x \cos y} = -ctg x \sin y \, dy \quad /\cdot \frac{\cos y}{ctg x}$$

$$\frac{dx}{\cos^2 x \, ctg x} = -\cos y \sin y \, dy$$

ovo je diferencijalna jednačina sa razdvojenim promjenjivim

$$\frac{tg x}{\cos^2 x} dx + \sin y \cos y \, dy = 0 \quad /\int$$

$$\int tg x \, d(tg x) + \int \sin y \, d(\sin y) = C_1$$

$$\frac{1}{2}tg^2x+\frac{1}{2}\sin^2y=\frac{1}{2}\,c$$

$$tg^2x+\sin^2y=C\,.$$ opšte rješenje diferencijalne jednačine

c) $$(\sqrt{xy}+\sqrt{x})\,y'-y=0$$

$$(\sqrt{y}+1)\sqrt{x}\;y'=y$$

$$y'=\frac{1}{\sqrt{x}}\;\frac{y}{\sqrt{y}+1}$$ ovo je diferencijalna jednačina sa razdvojenim promjenljivim

$$y'=\frac{dy}{dx}\qquad \frac{dy}{dx}=\frac{1}{\sqrt{x}}\;\frac{y}{\sqrt{y}+1}$$

$$(\sqrt{y}+1)\sqrt{x}\,\frac{dy}{dx}=y\;\Rightarrow\;\frac{\sqrt{y}+1}{y}\,dy=\frac{1}{\sqrt{x}}\,dx$$

$$\frac{\sqrt{y}+1}{y}\,dy-\frac{1}{\sqrt{x}}\,dx=0\qquad /\!/\int$$

$$\int\left(y^{-1/2}+\frac{1}{y}\right)dy-\int x^{-\frac{1}{2}}\,dx=C$$

$$2\sqrt{y}+\ln|y|-2\sqrt{x}=C$$ opšte rješenje diferencijalne jednačine

d) $$2^{x+y}+3^{x-2y}\,y'=0$$

$$2^x\cdot2^y+3^x\cdot3^{-2y}\,\frac{dy}{dx}=0\qquad /\cdot\frac{dx}{2^y3^x}$$

primjetimo da je riječ o diferencijalnoj jednačini sa razdvojenim promjenljivim

$$\frac{2^x}{3^x}\,dx+\frac{3^{-2y}}{2^y}\,dy=0\qquad /\!/\int$$

$$\int\left(\frac{2}{3}\right)^x dx+\int\left(\frac{1}{18}\right)^y dy=C\qquad \frac{\left(\frac{2}{3}\right)^x}{\ln\frac{2}{3}}-\frac{\left(\frac{1}{18}\right)^y}{\ln 18}=C$$

opšte rješenje diferencijalne jednačine

(#) Odrediti partikularno rješenje diferencijalne jednačine, koji zadovoljavaju inicijalni uslov:

a) $y\,dx + ctg\,x\,dy = 0;$ $y(\frac{\pi}{3}) = -1$

b) $s = s' \cos^2 t \ln s;$ $s(\pi) = 1.$

Rj. a) $y\,dx + ctg\,x\,dy = 0 \quad / \cdot \frac{1}{y\,ctg\,x}$

$$\frac{dx}{ctg\,x} + \frac{dy}{y} = 0 \quad /\int$$

$$\int \frac{\sin x}{\cos x}dx + \int \frac{dy}{y} = C_1$$

$$-\ln|\cos x| + \ln|y| = \ln C_2$$

$$\ln|y| = \ln C_2|\cos x|$$

$$|y| = C_2|\cos x|$$

$$y = \pm C_2 \cos x = C\cos x$$

$y = C\cos x$ je opšte rješenje diferencijalne jednačine

Da bi odredili partikularno rješenje trebamo odrediti konstantu $C$ tako da je $y(\frac{\pi}{3}) = -1$. Ovo znači da je $y = -1$, $x = \frac{\pi}{3}$ u opštem rješenju diferencijalne jednačine.

$$-1 = C \cdot \underbrace{\cos(\tfrac{\pi}{3})}_{=\frac{1}{2}} \Rightarrow C = \frac{-1}{\frac{1}{2}} = -2$$

$y = -2\cos x$ je partikularno rješenje diferencijalne jednačine

b) $s = s' \cos^2 t \ln s$

$$s = \frac{ds}{dt} \cos^2 t \ln s \quad / \frac{dt}{\cos^2 t} \cdot \frac{1}{s}$$

$$\frac{dt}{\cos^2 t} = \frac{\ln s}{s} ds \quad / \int$$

$$\int d(\operatorname{tg} t) = \int \ln s \, d(\ln s) + C$$

$\operatorname{tg} t = \frac{1}{2} \ln^2 s + C$ opšte rješenje diferencijalne jednačine

Sad ako stavimo da je $t = \pi$, $s = 1$ imam

$\underbrace{\operatorname{tg} \pi}_{=0} = \frac{1}{2} \underbrace{\ln^2 1}_{=0} + C \Rightarrow$ $C = 0$ je partikularno rješenje diferencijalne jednačine

**Diferencijalne jednačine sa razdvojenim promjenjivim su oblika y'=f(x)g(y).**

① Riješiti diferencijalnu jednačinu $xy' = y - xy\sin x$.

Rj. $xy' = y - xy\sin x$

$xy' = y(1 - x\sin x) \quad /: x \ (x \neq 0)$

$y' = y \cdot \frac{1 - x\sin x}{x}$ ovo je dif. jedn. sa razdv. promj.

$\frac{dy}{dx} = y \cdot \frac{1 - x\sin x}{x} \quad / \cdot \frac{dx}{y}$

$\frac{dy}{y} = \left(\frac{1}{x} - \sin x\right) dx \quad /\int$

$\int \frac{dy}{y} = \int \frac{1}{x} dx - \int \sin x \, dx$

$\ln|y| = \ln|x| + \cos x + \ln C$

$\ln|y| = \ln|x \cdot c| + \ln e^{\cos x}$

$y = Cx e^{\cos x}$ opšte rješenje dif. jedn.

ISPITNI ZADATAK

② Riješiti diferencijalnu jednačinu

$(xy^2 + 3x)dx + (2x^2y - 5y)dy = 0.$

Rj. $(2x^2y - 5y)dy = -(xy^2 + 3x)dx$

$y(2x^2 - 5)dy = -x(y^2 + 3)dx$

$\frac{y}{y^2+3} dy = \frac{-x}{2x^2 - 5} dx$ ovo je dif. jedn. sa razdv. prom.

$\int \frac{y}{y^2+3} dy = -\int \frac{x}{2x^2-5} dx$

$\frac{1}{2} \ln|y^2+3| = -\frac{1}{4} \ln|2x^2 - 5| + \ln C_1 \quad / \cdot 4$

$\ln|y^2+3|^2 = \ln|C \cdot (2x^2-5)^{-1}|$

$\int \frac{y}{y^2+3} dy = \left| \begin{matrix} y^2 = t \\ 2ydy = dt \end{matrix} \right| = \frac{1}{2} \int \frac{dt}{t+3}$

$= \frac{1}{2} \ln|y^2+3|$

$\int \frac{x}{2x^2-5} dx = \left| \begin{matrix} 2x^2 = t \\ 4xdx = dt \\ xdx = \frac{1}{4} dt \end{matrix} \right| = \frac{1}{4} \int \frac{dt}{t-5}$

$= \frac{1}{4} \ln|2x^2-5|$

$(y^2+3)^2 = \frac{C}{2x^2-5}$

opšte rješenje dif. jedn.

③ ✓ Riješiti diferencijalnu jednačinu

$3y'(x^2-1) - 2xy = 0$

Rj. $y^3 = C(x^2-1)$ opšte rješ. dif. jedn.

(#) Riješiti diferencijalnu jednačinu $y-xy'=a(1+x^2y')$, $a=const.$

Rj. $y-xy'=a(1+x^2y')$, $a=const.$

$$y-xy'=a+ax^2y'$$

$$ax^2y'+xy'=y-a$$

$$(ax^2+x)y'=y-a$$

$$y'=\frac{1}{ax^2+x}\cdot(y-a)$$

Ovo je diferenc. jednačina sa razdvojenim promjenjivim

$$y'=\frac{dy}{dx}$$

$$\frac{dy}{y-a}=\frac{dx}{ax^2+x}$$

$$\int\frac{dx}{x(ax+1)}=\int\frac{dy}{y-a}$$

$$ax+1=t$$
$$adx=dt$$
$$dx=\frac{1}{a}dt$$

$$\uparrow$$

$$\int\frac{dx}{x(ax+1)}=\int\left(\frac{1}{x}-\frac{a}{ax+1}\right)dx$$

$$=\ln|x|-a\cdot\frac{1}{a}\ln|ax+1|+c$$

$$=\ln\left|\frac{x}{ax+1}\right|+c$$

$$\ln\left|\frac{x}{ax+1}\right|=\ln|y-a|+\ln c$$

$$\frac{x}{ax+1}=c(y-a)$$

Rešenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu $(x^2y+x^2)dx+(x^4y-y)dy=0$.

Rj: $(x^2y+x^2)dx+(x^4y-y)dy=0$

$$x^2(y+1)dx+(x^4-1)y\,dy=0$$

$$x^2(y+1)dx=-(x^4-1)y\,dy$$

$$\frac{y}{y+1}dy=-\frac{x^2}{x^4-1}dx$$

diferencijalni račun sa razdvojenim promjenjivim //

$$\int\frac{y}{y+1}dy=-\int\frac{x^2}{x^4-1}dx$$

$$\int\frac{y^{+1-1}}{y+1}dy=\int dy-\int\frac{dy}{y+1}=y-\ln|y+1|+c$$

$$\int\frac{x^2}{x^4-1}=\frac{x^2}{(x-1)(x+1)(x^2+1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x^2+1}\quad/(x^4-1)$$

$$x^2=A\underbrace{(x+1)(x^2+1)}_{x^3+x+x^2+1}+B\underbrace{(x-1)(x^2+1)}_{x^3+x-x^2-1}+C\underbrace{(x-1)(x+1)}_{x^2+x-x-1}$$

$$x^2=A(x^3+x^2+x+1)+B(x^3-x^2+x-1)+C(x^2-1)$$

$A+B=0$ (a) $\quad A=-B$

$A-B+C=1$ (b) $\quad$ (b): $-B-B+C=1 \Rightarrow -2B+C=1$ } $+ \Rightarrow -4B=1$

$A+B=0$ (c) $\quad$ (d): $-B-B-C=0 \Rightarrow -2B-C=0$ } $\quad B=-\frac{1}{4}$

$A-B-C=0$ (d)

$\Rightarrow A=\frac{1}{4} \qquad \frac{1}{4}+\frac{1}{4}+C=1 \Rightarrow C=\frac{1}{2}$

$$\int\frac{x^2}{x^4-1}dx=\frac{1}{4}\int\frac{dx}{x-1}-\frac{1}{4}\int\frac{dx}{x+1}+\frac{1}{2}\int\frac{dx}{x^2+1}=\frac{1}{4}\ln|x-1|-\frac{1}{4}|x+1|+\frac{1}{2}\operatorname{arctg}x+C$$

$$y-\ln|y+1|=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|+\frac{1}{2}\operatorname{arc\,tg}x+C$$

rješenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu $y' = 2^{2x+y}$.

Rj. $y' = 2^{2x} \cdot 2^{y}$ diferencijalna jednačina sa razdvojenim promjenjivim

$$\frac{dy}{dx} = 2^{2x} \cdot 2^{y}$$

$$\frac{dy}{2^{y}} = 4^{x}\,dx$$

$$\int a^{x}dx = \frac{a^{x}}{\ln a} + C, \quad 0 < a \neq 1$$

$$2^{-y}dy = 4^{x}\,dx \quad //$$

$$\int 2^{-y}dy = \int 4^{x}dx$$

$$\ln 4 = \ln 2^{2} = 2\ln 2$$

$$-\frac{2^{-y}}{\ln 2} = \frac{4^{x}}{\ln 4} + C_1$$

$$\int 2^{-y}dy = \left|\begin{matrix} -y = t \\ -dy = dt \\ dy = -dt \end{matrix}\right| = -\int 2^{t}dt$$

$$-\frac{2^{-y}}{\ln 2} = \frac{4^{x}}{2\ln 2} + C_1 \quad / \cdot \ln 2 \cdot 2$$

$$= -\frac{2^{t}}{\ln 2} + C = -\frac{2^{-y}}{\ln 2} + C$$

$$-2 \cdot 2^{-y} = 4^{x} + C$$ opšte rješenje dif. jednačine

$$2^{-y} = \frac{4^{x} + C}{-2}$$ ili $$4^{x} = C - 2 \cdot 2^{-y}$$ ili

$$-y = \log_2 \frac{4^{x} + C}{-2}$$ $$x = \log_4 (C - 2^{1-y})$$ opšte rješenje

$$y = \log_2 \frac{-2}{4^{x} + C}$$ opšte rješenje

(#) Riješiti diferencijalnu jednačinu

$$x^2(y+1)\,dx+y^2(x-1)\,dy=0 .$$

Rj.

Prisjetimo se: Za varijable jednačine $M(x,y)dx+N(x,y)dy=0$ kažemo da su razdvojive ako se jednačina može napisati u obliku

$$f_1(x)\cdot g_2(y)\,dx+f_2(x)\cdot g_1(y)\,dy=0$$

Ako ovu jednakost pomnožimo sa $\frac{1}{f_2(x)\,g_2(x)}$ dobićemo

$$\frac{f_1(x)}{f_2(x)}\,dx+\frac{g_1(y)}{g_2(y)}\,dy=0$$

iz čega integraljenjem možemo dobiti primitivnu f-ju.

$x^2(y+1)\,dx+y^2(x-1)\,dy=0$ ovo je dif. jedn. sa razdvojenim promjenjivim

$$/\cdot\frac{1}{(y+1)(x-1)}$$

$$\frac{x^2}{x-1}\,dx+\frac{y^2}{y+1}\,dy=0$$

$$\begin{array}{l} x^2:(x-1)=x+1 \\ \underline{-\ x^2-x} \\ \qquad x \\ \underline{-\ x-1} \\ \qquad +1 \end{array}$$

$$\left(x+1+\frac{1}{x-1}\right)dx+\left(y-1+\frac{1}{y+1}\right)dy=0 \qquad \int\int$$

$$\begin{array}{l} y^2:(y+1)=y-1 \\ \underline{-\ y^2+y} \\ \qquad -y \\ \underline{-\ -y-1} \\ \qquad +1 \end{array}$$

$$\frac{1}{2}x^2+x+\ln(x-1)+\frac{1}{2}y^2-y+\ln(y+1)=C_1 \;/\cdot 2$$

$$x^2+y^2+2x-2y+2\ln(x-1)(y+1)=C_2$$

$$x^2+2\cdot x\cdot 1+1-1+y^2-2\cdot y\cdot 1+1-1+2\ln(x-1)(y+1)=C_2$$

$$(x+1)^2+(y-1)^2+2\ln(x-1)(y+1)=C$$ traženo rješenje

opšte rješ. dif. jedn.

(#) Riješiti diferencijalnu jednačinu

$$4x\,dy - y\,dx = x^2\,dy\,.$$

Rj.

$$4x\,dy - y\,dx = x^2\,dy \quad /\cdot(-1)$$

$$y\,dx + (x^2-4x)\,dy = 0$$

ovo je diferencijalna jednačina sa razdvojenim promjenjivim

$$/\cdot\frac{1}{(x^2-4x)y}$$

$$\frac{dx}{x^2-4x} + \frac{dy}{y} = 0$$

$$\frac{1}{x^2-4x} = \frac{1}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4} \quad /\cdot(x^2-4x)$$

$$1 = A(x-4) + Bx \quad \Rightarrow \quad \begin{array}{l} A+B=0 \\ -4A \quad =1 \end{array} \qquad A=-\frac{1}{4} \quad B=\frac{1}{4}$$

$$\left(\frac{-\frac{1}{4}}{x} + \frac{\frac{1}{4}}{x-4}\right)dx + \frac{1}{y}\,dy = 0 \quad /\int$$

$$-\frac{1}{4}\ln x + \frac{1}{4}\ln(x-4) + \ln y = C_1 \quad /\cdot 4$$

$$-\ln x + \ln(x-4) + 4\ln y = \ln C$$

$$\ln\frac{x-4}{x} + \ln y^4 = \ln C$$

$$(x-4)y^4 = Cx$$

opšte rj. dif. jedn.
traženo rješenje

(#) Riješiti diferencijalnu jednačinu

$$\frac{dy}{dx} = \frac{4y}{x(y-3)}$$

Rj:

$$\frac{dy}{dx} = \frac{4y}{x(y-3)} \quad \Big/ \cdot dx \cdot \frac{y-3}{y}$$

$$\frac{y-3}{y}\,dy = \frac{4\,dx}{x}$$

ovo je diferencijalna jednač. sa razdvojenim promjenjivim

$$\left(1 - \frac{3}{y}\right) dy = \frac{4}{x}\,dx \quad \int\int$$

$$y - 3\ln y = 4\ln x + \ln C_1$$

$$y = \ln x^4 + 3\ln y + \ln C_1$$

$$y = \ln\left(C_1 x^4 y^3\right)$$

$$C_1 x^4 y^3 = e^y$$

$$x^4 y^3 = Ce^y$$

opšte rješenje dif. jedn.
traženo
rješenje

(#) Odrediti partikularno rješenje diferencijalne jednačine $(1+x^3)dy - x^2 y\,dx = 0$ koje zadovoljava inicijalni uslov $x=1$, $y=2$.

Rj.

$(1+x^3)dy - x^2 y\,dx = 0$ ovo je diferencijalna jednačina sa razdvojenim promjenjivim

$$\Big/ \frac{1}{y(1+x^3)}$$

$$\frac{dy}{y} - \frac{x^2}{1+x^3}\,dx = 0 \qquad //\int$$

$$\int\frac{dy}{y} - \frac{1}{3}\int\frac{d(1+x^3)}{1+x^3} = C_1$$

$$\ln y - \frac{1}{3}\ln(1+x^3) = \ln C_2 \quad /\cdot 3$$

$$3\ln y = \ln C(1+x^3)$$

$y^3 = C(1+x^3)$ opšte rješenje dif. jedn.

Za $x=1$, $y=2$: $\quad 2^3 = C(1+1) \Rightarrow C=4$

$y^3 = 4(1+x^3)$ partikularno rješenje diferencijalne jednačine

① Riješiti diferencijalnu jednačinu $xy'+y=-x$.

Rj. $xy'+y=-x \quad /:x \ (x\neq 0)$

$y'+\frac{y}{x}=-1$

$y'=-1-\frac{y}{x}$ ovo je hom. dif. jedn.

uvodimo smjenu $u=\frac{y}{x}$

$y=ux, \ y'=u'x+u$

$u'x+u=-1-u$

$\frac{du}{dx}x=-1-2u \quad /\cdot\frac{dx}{(-1-2u)\cdot x}$

$\frac{du}{-1-2u}=\frac{dx}{x}$

$\frac{du}{2u+1}=-\frac{dx}{x} \quad \int\int$ $\quad \left| \begin{matrix} 2u=t \\ 2du=dt \\ du=\frac{1}{2}dt \end{matrix} \right|$

$\frac{1}{2}\ln|2u+1|=-\ln|x|+\ln|C_1| \quad /\cdot 2$

$\ln|2u+1|=2\ln\left|\frac{C_1}{x}\right|$

$2u+1=\left(\frac{C_1}{x}\right)^2$

$2u=\frac{C}{x^2}-1 \Rightarrow 2\frac{y}{x}=\frac{C}{x^2}-1 \Rightarrow 2y=\frac{C}{x^2}-x$ tj. $y=\frac{C}{x^2}-\frac{x}{2}$

opšte rješenje diferenc. jedn.

② Nađi partikularno rješenje diferencijalne jednačine $xy'=y(1+\ln y-\ln x)$ tako da zadovoljava uslov $y(1)=e$.

Rj. $xy'=y(1+\ln\frac{y}{x}) \quad /:x$

$y'=\frac{y}{x}(1+\ln\frac{y}{x})$ ovo je hom. dif. jedn.

$u=\frac{y}{x} \Rightarrow y=ux, \ y'=u'x+u$

$u'x+u=u(1+\ln u)$

$u'x=u\ln u, \quad u'=\frac{du}{dx}$

$\frac{du}{u\ln u}=\frac{dx}{x} \quad \int\int$

$\int\frac{du}{u\ln u}=\left|\begin{matrix}\ln u=t \\ \frac{du}{u}=dt\end{matrix}\right|=\int\frac{dt}{t}=\ln|t|$
$=\ln\ln u$

$\ln\ln u=\ln x+\ln C$

$\ln u=xC \Rightarrow u=e^{cx}$

$y=xe^{cx}$ opšte rješenje dif. jedn.

$\left.\begin{matrix} y(1)=e \\ y(1)=1\cdot e^{c\cdot 1}\end{matrix}\right\} \Rightarrow e^c=e \Rightarrow c=1$

$y=xe^x$ partikularno rješenje dif. jedn.

③ Nađi opšte rješenje dif. jednačine $xy'=xe^{\frac{y}{x}}+y$.

Rj. $y=-x\ln\ln\frac{C}{x}$

④ Riješiti diferencijalnu jednačinu

$$y^3y' + 3xy^2 + 2x^3 = 0.$$

Rj.

$$y^3y' + 3xy^2 + 2x^3 = 0$$

$$y^3y' = -3xy^2 - 2x^3 \quad /:y^3$$

$$y' = \frac{-3xy^2 - 2x^3}{y^3} \quad \begin{matrix} :x^3 \\ :x^3 \end{matrix}$$

$$y' = \frac{-3\left(\frac{y}{x}\right)^2 - 2}{\left(\frac{y}{x}\right)^3}$$

ovo je homogena diferencijalna jednačina

uvodimo smjenu $\frac{y}{x} = u$

tj. $y = ux$

$y' = u'x + u$

$$u'x + u = \frac{-3u^2 - 2}{u^3}$$

$$u'x = \frac{-3u^2 - 2}{u^3} - u$$

$$u'x = \frac{-3u^2 - 2 - u^4}{u^3}$$

$$\frac{du}{dx}x = \frac{-u^4 - 3u^2 - 2}{u^3}$$

$$\frac{u^3}{-u^4 - 3u^2 - 2}du = \frac{dx}{x}$$

$$\frac{u^3}{u^4 + 3u^2 + 2}du = -\frac{dx}{x}$$

$$u^4 + 3u^2 + 2 = 0$$

$u^2 = t$, $\quad t^2 + 3t + 2 = 0$ $\qquad t_{1,2} = \frac{-3 \pm 1}{2}$

$D = 9 - 8 = 1$ $\qquad t_1 = \frac{-4}{2} = -2$

$(u^2+2)(u^2+1) = 0$ $\qquad t_2 = \frac{-2}{2} = -1$

$$\frac{u^3}{u^4 + 3u^2 + 2} = \frac{Au + B}{u^2 + 2} + \frac{Cu + D}{u^2 + 1} \quad /(u^2+2)(u^2+1)$$

$$u^3 = A(\underset{=}{u^3} + \underset{\equiv}{u}) + B(\underline{u^2+1}) + C(\underset{=}{u^3} + \underset{\equiv}{2u}) + D(\underline{u^2+2})$$

$$\begin{aligned} A + \quad C \quad\quad &= 1 \\ B \quad\quad + D &= 0 \\ A + \quad 2C \quad\quad &= 0 \\ B \quad\quad + 2D &= 0 \end{aligned}$$

---

$$B = D = 0$$

$A + C = 1$

$A + 2C = 0 \quad / \cdot (-1)$ $\qquad \therefore A = 2$

$A + C = 1$

$-A - 2C = 0$

$-C = 1$

$C = -1$

$$\frac{u^3}{u^4+3u^2+2} = \frac{2u}{u^2+2} + \frac{-u}{u^2+1}$$

$$\frac{u^3}{u^4+3u^2+2}\, du = - \frac{dx}{x} \quad / \int$$

$$\ln|u^2+2| - \frac{1}{2}\ln|u^2+1| = -\ln|x| + \ln c$$

$$\ln \frac{|u^2+2|}{\sqrt{u^2+1}} = \ln \frac{c}{x}$$

$$\frac{u^2+2}{\sqrt{u^2+1}} = \frac{c}{x}$$

$$\frac{\left(\frac{y}{x}\right)^2+2}{\sqrt{\left(\frac{y}{x}\right)^2+1}} = \frac{c}{x}$$

rješenje
diferencijalne
jednačine

(#) Riješiti diferencijalnu jednačinu

$$(3y^2+3xy+x^2)dx=(x^2+2xy)dy$$

Rj:

$$(x^2+2xy)dy=(3y^2+3xy+x^2)dx \qquad /:dx \quad /:(x^2+2xy)$$

$$\frac{dy}{dx}=\frac{3y^2+3xy+x^2}{x^2+2xy}\begin{matrix}:x^2\\ :x^2\end{matrix}$$

$$y'=\frac{3\left(\frac{y}{x}\right)^2+3\frac{y}{x}+1}{2\frac{y}{x}+1}$$

ovo je homogena difer. jedn. uvodimo smjenu $u=\frac{y}{x}$

$$y=ux \quad /\frac{d}{dx}$$

$$y'=u'x+u$$

$$u'x+u=\frac{3u^2+3u+1}{2u+1}$$

$$u'x=\frac{3u^2+3u+1}{2u+1}-u$$

$$u'x=\frac{3u^2+3u+1-2u^2-u}{2u+1}$$

$$u'x=\frac{u^2+2u+1}{2u+1}$$

$$\frac{du}{dx}x=\frac{u^2+2u+1}{2u+1}$$

$$\frac{2u+1}{u^2+2u+1}du=\frac{dx}{x} \quad \dots(*)$$

$$\int\frac{2u+1}{u^2+2u+1}du=\int\frac{2u+2-1}{u^2+2u+1}du=$$

$$=\int\frac{2u+2}{u^2+2u+1}du-\int\frac{du}{u^2+2u+1}=$$

$$=\left|\begin{matrix}u^2+2u+1=t\\ (2u+2)du=dt\end{matrix}\right|=\int\frac{dt}{t}-\int\frac{du}{(u+1)^2}=\left|\begin{matrix}u+1=s\\ du=ds\end{matrix}\right|=$$

$$\ln|t|-\int\frac{ds}{s^2}=\ln|u^2+2u+1|-\frac{s^{-1}}{(-1)}+c=\ln(u+1)^2+\frac{1}{u+1}+c$$

$$(*)\Rightarrow \ln(u+1)^2+\frac{1}{u+1}=\ln|x|+c$$

$$\ln\left(\frac{y}{x}+1\right)^2+\frac{1}{\frac{y}{x}+1}=\ln|x|+c$$

rješenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu

$$(5y+7x)\,dy+(8y+10x)\,dx=0$$

Rj:

$$(5y+7x)\,dy+(8y+10x)\,dx=0$$

$$(5y+7x)\,dy=(-8y-10x)\,dx=0$$

$$\frac{dy}{dx}=\frac{-8y-10x \quad /:x}{5y+7x \quad /:x}$$

$$y'=\frac{-8\left(\frac{y}{x}\right)-10}{5\left(\frac{y}{x}\right)+7}$$

ovo je homogena diferencijalna jednačina, uvodimo smjenu $u=\frac{y}{x}$

$$y=u\cdot x \quad \Big/\frac{d}{dx}$$

$$y'=u'x+u$$

$$u'x+u=\frac{-8u-10}{5u+7}$$

$$u'x=\frac{-8u-10}{5u+7}-u$$

$$u'x=\frac{-8u-10-u(5u+7)}{5u+7}$$

$$u'x=\frac{-8u-10-5u^2-7u}{5u+7}$$

$$\frac{du}{dx}x=\frac{\overbrace{-5u^2-15u-10}^{(-5)(u^2+3u+2)}}{5u+7}$$

$$\frac{du}{dx}x=(-5)\frac{(u+1)(u+2)}{5u+7}$$

$$\frac{(5u+7)\,du}{u^2+3u+2}=-5\,\frac{dx}{x} \quad \dots(*) \quad \Big/\int$$

$$\frac{5u+7}{u^2+3u+2}=\frac{5u+7}{(u+1)(u+2)}=\frac{A}{u+1}+\frac{B}{u+2} \quad /(u+1)(u+2)$$

$$5u+7=A(u+2)+B(u+1)$$

$$A+B=5$$

$$-\ 2A+B=7$$

$$-A=-2$$

$$A=2$$

$$B=3$$

$$\int\frac{5u+7}{u^2+3u+2}\,du=2\int\frac{du}{u+1}+3\int\frac{du}{u+2}$$

$$(*)\ \Rightarrow\ 2\ln|u+1|+3\ln|u+2|=-5\ln|x|+\ln|C|$$

$$\ln(u+1)^2(u+2)^3=\ln(x^{-5}C)$$

$$(u+1)^2(u+2)^3=\frac{C}{x^5}$$

$$\left(\frac{y}{x}+1\right)^2\left(\frac{y}{x}+2\right)^3=\frac{C}{x^5}$$

rješenje diferencijalne jednačine

(#) Odrediti opšte rješenje date diferencijalne jednačine

$$y' = \frac{x+y}{x-y}.$$

Rj.:

$$y' = \frac{x+y}{x-y} \quad \begin{matrix} /:x \\ /:x \end{matrix}$$

$$y' = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$$

ovo je homogena diferencijalna jednačina $y' = f(\frac{y}{x})$, uvodimo smjenu $\frac{y}{x} = u$, $y = ux$, $y' = u'x + u$

$$u'x + u = \frac{1+u}{1-u}$$

$$u'x = \frac{1+u}{1-u} - u$$

$$u' = \frac{du}{dx}, \quad \frac{du}{dx}x = \frac{1+\overbrace{u-u(1-u)}^{-u+u^2}}{1-u}$$

$$\frac{1-u}{1+u^2}du = \frac{dx}{x} \quad /\int$$

$$\int \frac{du}{1+u^2} - \frac{1}{2}\int \frac{d(1+u^2)}{1+u^2} = \int \frac{dx}{x}$$

$$\operatorname{arctg} u - \frac{1}{2}\ln|1+u^2| = \ln|x| + \ln C$$

$$1+u^2 = 1+\frac{y^2}{x^2} = \frac{x^2+y^2}{x^2}$$

$$\operatorname{arctg}\frac{y}{x} = \ln|xC| + \ln\frac{\sqrt{x^2+y^2}}{x}$$

$$\operatorname{arctg}\frac{y}{x} = \ln C\sqrt{x^2+y^2}$$ opšte rješenje diferencijalne jednačine

(#) Odrediti opšte rješenje date diferencijalne jednačine

$$y' = \frac{y^2}{x^2} - 2 .$$

Rj. $y' = \left(\frac{y}{x}\right)^2 - 2$ ovo je homogena diferencijalna jednačina $y' = f\left(\frac{y}{x}\right)$, uvodimo smjenu $\frac{y}{x} = u$, $y = ux$, $y' = u'x + u$.

$$y' = u^2 - 2$$

$$u'x + u = u^2 - 2$$

$$u'x = u^2 - u - 2$$

$$u' = \frac{du}{dx}$$

$$\frac{du}{dx} = \frac{u^2 - u - 2}{x}$$

$$\frac{du}{u^2 - u - 2} = \frac{dx}{x}$$

$$u^2 - u - 2 = (u-2)(u+1)$$

$$\frac{1}{(u-2)(u+1)} = \frac{A}{u-2} + \frac{B}{u+1}$$

$$1 = A(u+1) + B(u-2)$$

$$A + B = 0$$

$$- \quad A - 2B = 1$$

$$3B = -1$$

$$B = \frac{-1}{3} \qquad A = \frac{1}{3}$$

$$\int \left( \frac{\frac{1}{3}}{u-2} - \frac{\frac{1}{3}}{u+1} \right) du = \int \frac{dx}{x}$$

$$\frac{1}{3} \ln|u-2| - \frac{1}{3} \ln|u+1| = \ln|x| + \ln C_1$$

$$\ln \frac{\sqrt[3]{|u-2|}}{\sqrt[3]{|u+1|}} = \ln|x| C_1$$

$$\frac{\sqrt[3]{|u-2|}}{\sqrt[3]{|u+1|}} = |x| C_1$$

$$\frac{u-2}{u+1} = x^3 C$$

vratimo smjenu $\dfrac{\frac{y}{x} - 2}{\frac{y}{x} + 1} = x^3 C$

$$\frac{y - 2x}{y + x} = x^3 C$$

$y - 2x = Cx^3(y + x)$ opšte rješenje diferencijalne jednačine

㊕ Odrediti opšte rješenje date diferencijalne jednačine

$$x\,dy - y\,dx = y\,dy .$$

Rj:

$$x\,dy - y\,dx = y\,dy \quad /: dx$$

$$x\frac{dy}{dx} - y = y\frac{dy}{dx}$$

$$\frac{dy}{dx} = y'$$

$$xy' - yy' = y$$

$$(x-y)y' = y$$

$$y' = \frac{y \;/:x}{x-y \;/:x}$$

$$y' = \frac{\frac{y}{x}}{1-\frac{y}{x}}$$

ovo je homogena diferencijalna jednačina $y' = f(\frac{y}{x})$, uvodimo smjenu $\frac{y}{x} = u \Rightarrow$

$y = ux$, $y' = u'x + u$

$$u'x + u = \frac{u}{1-u}$$

$$u'x = \frac{u}{1-u} - u$$

$$u' = \frac{du}{dx}, \quad \frac{du}{dx}x = \frac{\overset{u-u+u^2}{u-u(1-u)}}{1-u}$$

$$\frac{du}{dx}x = \frac{u^2}{1-u}$$

$$\frac{1-u}{u^2}du = \frac{dx}{x}$$

$$\left(\frac{1}{u^2} - \frac{1}{u}\right)du = \frac{dx}{x}$$

$$\frac{u^{-1}}{-1} - \ln u = \ln|x| + c_1$$

$$-\frac{1}{u} - \ln u = \ln|x| + c_1$$

vratimo smjenu $u = \frac{y}{x}$

$$-\frac{1}{\frac{y}{x}} - \ln\frac{|y|}{|x|} = \ln x + c_1$$

$$-\frac{x}{y} - \ln|y| + \ln|x| = \ln|x| + c_1 \quad /\cdot(-1)$$

$$\frac{x}{y} + \ln|y| = c$$

je opšte rješenje date diferencijalne jednačine

(#) Odrediti opšte rješenje date diferencijalne jednačine

$$y' = \frac{2xy}{x^2-y^2}.$$

Rj. $$y' = \frac{2xy}{x^2-y^2}\begin{array}{l} /:x^2 \\ /:x^2 \end{array}$$

$$y' = \frac{2\frac{y}{x}}{1-\left(\frac{y}{x}\right)^2}$$

ovo je homogena diferencijalna jednačina $y' = f(\frac{y}{x})$, uvodimo smjenu $\frac{y}{x} = u$, $y = ux$, $y' = u'x + u$

$$u'x + u = \frac{2u}{1-u^2}$$

$$u'x = \frac{2u}{1-u^2} - u$$

$$u' = \frac{du}{dx}$$

$$\frac{du}{dx} = \frac{2u\overbrace{-u(1-u^2)}^{-u+u^3}}{1-u^2}\cdot\frac{1}{x}$$

$$\frac{1-u^2}{u+u^3}\,du = \frac{dx}{x} \quad \int\int$$

$$\frac{1-u^2}{u(1+u^2)} = \frac{A}{u} + \frac{Bu+C}{1+u^2} \quad /\cdot u(1+u^2)$$

$$1-u^2 = A + Au^2 + Bu^2 + Cu$$

$A + B = -1 \Rightarrow B = -2$

$C = 0$

$A = 1$

$$\int\left(\frac{1}{u} - \frac{2u}{1+u^2}\right)du = \int\frac{dx}{x}$$

$$\int\frac{du}{u} - \int\frac{d(1+u^2)}{1+u^2} = \int\frac{dx}{x}$$

$$\ln|u| - \ln|1+u^2| = \ln|x| + \ln|C_1|$$

$$\ln\frac{u}{1+u^2} = \ln xC_1$$

$$\frac{u}{1+u^2} = xC_1$$

ako vratimo smjenu $u = \frac{y}{x}$

$$\frac{\frac{y}{x}}{1+\frac{y^2}{x^2}} = xC_1$$

$$\frac{\frac{y}{x}}{\frac{x^2+y^2}{x^2}} = xC_1$$

$$\frac{y}{x^2+y^2}\cdot x = xC_1 \quad /:C_1 : x \quad /\cdot(x^2+y^2)$$

$$x^2 + y^2 = Cy$$ opšte rješenje diferencijalne jednačine

# Diferencijalne jednačine koje se svode na homogene

su oblika $y' = f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$

ako je $a_1b_2-a_2b_1=0$ uvodimo smjenu $a_1x+b_1y=u$ i dobijamo dif. jedn. sa razdvojenim promjenjivim.

ako je $a_1b_2-a_2b_1\neq 0$ uvodimo smjenu $x=u+\alpha$, $y=v+\beta$ gdje $\alpha$ i $\beta$ dobijamo iz sistema

$$a_1\alpha+b_1\beta+c_1=0$$
$$a_2\alpha+b_2\beta+c_2=0 .$$

① Riješiti diferencijalnu jednačinu $(x-2y+1)y'=2x-y+1$.

Rj. $y'=\frac{2x-y+1}{x-2y+1}$, $a_1b_2-a_2b_1\neq 0 \Rightarrow x=u+\alpha,\ y=v+\beta$

$\Downarrow$

$y'=v'$

$$\left.\begin{array}{l}2\alpha-\beta+1=0\\ \alpha-2\beta+1=0\end{array}\right\} \Rightarrow \alpha=-\frac{1}{3} \text{ i } \beta=\frac{1}{3}$$

$x=u-\frac{1}{3}$

$y=v+\frac{1}{3}$

$$v'=\frac{2(u-\frac{1}{3})-(v+\frac{1}{3})+1}{(u-\frac{1}{3})-2(v+\frac{1}{3})+1}$$

$v'=\frac{2u-v}{u-2v}$ $/:u$ $/:u$,

$v'=\frac{2-\frac{v}{u}}{1-2\frac{v}{u}}$ ovo je hom. dif. jedn.

smjena $\frac{v}{u}=z$, $v=uz$

$v'=z'u+z$

$z'u+z=\frac{2-z}{1-2z}$

$z'u=\frac{2(z^2-z+1)}{1-2z}$, $z'=\frac{dz}{du}$

$\frac{1-2z}{z^2-z+1}dz=2\frac{du}{u}$ $\big/\!\!\int$

$-\ln(z^2-z+1)=2\ln u+\ln C_1$

$\ln\frac{1}{z^2-z+1}=\ln C_1u^2$

$1=C_1u^2(z^2-z+1)$

$1=C_1u^2\left(\frac{v^2}{u^2}-\frac{v}{u}+1\right)$ $/:C_1$

$C=v^2-uv+u^2$

$C=(y-\frac{1}{3})^2-(x+\frac{1}{3})(y-\frac{1}{3})+(x+\frac{1}{3})^2$

opšte rješenje diferenc. jednač.

② Riješiti diferencijalnu jednačinu

$(2x+y+1)y'=4x+2y+3$.

opšte rješenje

Rj. $\ln Cx^{16}(8x+4y+5)=$
$=4(2x+y+1)$

③ Riješiti diferencijalnu jednačinu

$(2x-4y+6)dx+(x+y-3)dy=0$

Rj. $(y-2x)^3=C(y-x-1)^2$

opšte rješenje

(#) Riješiti diferencijalnu jednačinu

$$(x-y-2)\,dx+(2x-y-5)\,dy=0$$

Rj.

$$(2x-y-5)\,dy=-(x-y-2)\,dx \quad \Big/\cdot\frac{1}{dx}\cdot\frac{1}{2x-y-5}$$

$$\frac{dy}{dx}=\frac{-x+y+2}{2x-y-5}$$

$$y'=\frac{-x+y+2}{2x-y-5}$$

diferencijalna jednačina koja se svodi na homogenu $y'=f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$

$$a_1b_2-a_2b_1=1-2=-1\neq 0$$

uvodimo smjenu $x=u+\alpha$, $y=v+\beta$

$$\begin{aligned}-\alpha+\beta+2&=0\\ +\ 2\alpha-\beta-5&=0\\ \hline \alpha-3&=0\\ \alpha&=3\end{aligned}$$

$$\begin{aligned}-\alpha+\beta+2&=0\\ -3+\beta+2&=0\\ \beta&=1\end{aligned}$$

$$x=u+3 \qquad y=v+1 \qquad y'=v' \qquad \Rightarrow \qquad u=x-3 \qquad v=y-1$$

$$v'=\frac{-u-3+v+1+2}{2u+6-v-1-5}$$

$$v'=\frac{-u+v}{2u-v}\ \begin{matrix}/:u\\ /:u\end{matrix}$$

$$v'=\frac{-1+\frac{v}{u}}{2-\frac{v}{u}}$$

ovo je homogena diferenc. jednačina

uvodimo smjenu $z=\frac{v}{u}$

$$v=z\cdot u \quad /'_u$$

$$v'=z'\cdot u+z$$

$$z'\cdot u+z=\frac{-1+z}{2-z}$$

$$z'u=\frac{-1+z}{2-z}-z$$

$$z'u=\frac{-1+z-2z+z^2}{2-z}$$

$$z'u=\frac{z^2-z-1}{2-z}, \quad z'=\frac{dz}{du}$$

$$\frac{2-z}{z^2-z-1}\,dz=\frac{du}{u} \quad /\!\int$$

$$2-z=(-1)(z-2)=\left(-\frac{1}{2}\right)(2z-4)=\left(-\frac{1}{2}\right)(2z-1-3$$

$$\int\frac{2-z}{z^2-z-1}\,dz=-\frac{1}{2}\int\frac{2z-1}{z^2-z-1}\,dz+\frac{3}{2}\int\frac{dz}{z^2-z-1}=$$

$$\overset{(*)}{=}-\frac{1}{2}\ln(z^2-z-1)+\frac{3\sqrt5}{10}\ln\left|\frac{2z-1-\sqrt5}{2z-1+\sqrt5}\right|+C$$

$$\frac{3}{2}\int\frac{dz}{z^2-z-1}=\left|\begin{matrix}z^2-z\ 1=z^2-2\cdot\frac{1}{2}\cdot z+\frac{1}{4}-\frac{1}{4}\ 1=\\ =\left(z-\frac{1}{2}\right)^2-\frac{5}{4}\end{matrix}\right|$$

$$=\frac{3}{2}\int\frac{dz}{\left(z-\frac{1}{2}\right)^2-\frac{5}{4}}=\left|\begin{matrix}z-\frac{1}{2}=\frac{\sqrt5}{2}t\\ dz=\frac{\sqrt5}{2}dt\\ t=\frac{2z-1}{\sqrt5}\end{matrix}\right|=\frac{3}{2}\cdot\frac{\sqrt5}{2}\cdot\frac{4}{5}\int\frac{dt}{t^2-1}$$

$$=\frac{3\sqrt5}{5}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C=\frac{3\sqrt5}{10}\ln\left|\frac{2z-1-\sqrt5}{2z-1+\sqrt5}\right| \ \ldots(*)$$

$$\ln u=-\frac{1}{2}\ln(z^2-z-1)+\frac{3\sqrt5}{10}\ln\left|\frac{2z-1-\sqrt5}{2z-1+\sqrt5}\right|+\ln C_2 \quad /\cdot 10$$

$$u^{10}=\frac{C}{(z^2-z-1)^5}\left|\frac{2z-1-\sqrt5}{2z-1+\sqrt5}\right|^{3\sqrt5}$$

opšte rješenje difer. jedn.

$$z=\frac{v}{u}, \quad v=y-1, \quad u=x-3$$

(#) Rješiti diferencijalnu jednačinu

$$(2x-5y+3)\,dx-(2x+4y-6)\,dy=0$$

Rj.

$$(2x+4y-6)\,dy=-(2x-5y+3)\,dx \quad /:dx \quad /:(2x+4y-6)$$

$$y'=\frac{-2x+5y-3}{2x+4y-6}$$ ovo je diferencijalna jednačina koja se svodi na homogenu

Tip smjene zavisi od vrijednosti $\begin{vmatrix} -2 & 5 \\ 2 & 4 \end{vmatrix}=-8-10=-18\neq 0$

⇒ Uvodimo smjenu $x=u+\alpha$, $y=v+\beta$ gdje brojeve $\alpha$ i $\beta$ dobijamo rješenjem sistema

$$-2\alpha+5\beta-3=0$$
$$2\alpha+4\beta-6=0$$

$\vdots$ $\alpha=1$, $\beta=1$

Uvodimo smjenu

$x=u+1 \Rightarrow dx=du$

$y=v+1 \Rightarrow dy=dv \quad y'=v'$

$$-2x+5y-3=-2u+5v$$
$$2x+4y-6=2u+4v$$

čime dobijamo sljedeću dif. jedn.

$$v'=\frac{-2u+5v}{2u+4v} \quad \begin{matrix}/:u\\ /:u\end{matrix}$$

$$v'=\frac{-2+5\frac{v}{u}}{2+4\frac{v}{u}}$$ ovo je homogena diferencijalna jednačina (prvog reda)

uvodimo smjenu $\frac{v}{u}=z$

$v=uz$

$v'=z'u+z$

$$z'u+z=\frac{-2+5z}{2+4z}$$

$\vdots$

$$\frac{du}{u}+\frac{4}{3}\cdot\frac{dz}{4z-1}+\frac{2}{3}\,\frac{dz}{z+2}=0$$

$$\ln u+\frac{1}{3}\ln(4z-1)+\frac{2}{3}\ln(z+2)=\ln C_1$$

$$u^3(4z-1)(z+2)^2=C$$

Mjenjajući $z$ sa $\frac{v}{u}$, $(4v-u)(v+2u)^2=C$

i mjenjajući $u$ sa $x-1$, i $v$ sa $y-1$

$(4y-x-3)(y+2x-3)^2=C$ traženo opšte rješenje dif. jedn.

(#) Rješiti diferencijalnu jednačinu

$$(x-y-1)\,dx+(4y+x-1)\,dy=0 .$$

Rj. Odmah primjetimo da je data diferencijalna jednačina koja se svodi na homogenu. Tip smjene zavisi od vrijednosti $\begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} = 1+4=5\neq 0$ ⇒ uvodimo smjenu

$$x=u+\alpha$$
$$y=v+\beta$$

gdje brojeve $\alpha$ i $\beta$ dobijamo rješenjem sistema

$$\alpha-\beta-1=0$$
$$\underline{4\beta+\alpha-1=0} \quad \ldots \quad \alpha=1, \quad \beta=0$$

uvodimo smjenu $x=u+1 \Rightarrow dx=du$, $y=v \Rightarrow dy=dv$ pa data jednačina se svodi na

$$(u-v)\,du+(4v+u)\,dv=0$$

a ovo je homogena dif. jednač. (step. 1)

uvodimo smjenu $v=zu$

$$dv=z\,du+u\,dz$$

iz čega dobijamo

$$(1-z)\,du+(4z+1)(z\,du+u\,dz)=0$$

$$\frac{du}{u}+\frac{4z+1}{4z^2+1}\,dz=\frac{du}{u}+\frac{1}{2}\,\frac{8z}{4z^2+1}\,dz+\frac{dz}{4z^2+1}=0$$

$$\ln u+\frac{1}{2}\ln(4z^2+1)+\frac{1}{2}\,\mathrm{arctg}\,2z=c_1 \Rightarrow \ln u^2(4z+1)+\mathrm{arctg}\,2z=c$$

$$\Rightarrow \ln(4v^2+u^2)+\mathrm{arctg}\,\frac{2v}{u}=c$$

$$\ln\left[4y^2+(x-1)^2\right]+\mathrm{arctg}\,\frac{2y}{x-1}=c$$

traženo opšte rješenje dif. jedn.

(#) Riješiti diferencijalnu jednačinu

$$(x+y)\,dx+(3x+3y-4)\,dy=0$$

Rj:

$$(3x+3y-4)\,dy=-(x+y)\,dx \quad /:dx \quad /:(3x+3y-4)$$

$$y'=\frac{-x-y}{3x+3y-4}$$ ovo je diferencijalna jednačina koja se svodi na homogenu

Tip smjene zavisi od vrijednosti $\begin{vmatrix} -1 & -1 \\ 3 & 3 \end{vmatrix} = -3+3=0.$

$\Rightarrow$ Uvodimo smjenu $x+y=u \Rightarrow$ $3x+3y=3u$

$y=u-x$

$dy=du-dx$

$$(x+y)\,dx+(2x+3y-4)\,dy=0$$

uvodimo naznačenu smjenu

$$u\,dx+(3u-4)(du-dx)=0$$

$$(4-2u)\,dx+(3u-4)\,du=0 \quad /:(4-2u)$$

$$dx+\frac{3u-4}{4-2u}\,du=0$$ ovo je diferencijalna jednačina sa razdvojenim promjenljivim

$$2\,dx+\frac{3u-4}{2-u}\,du=0 \Rightarrow 2\,dx-3\,du+\frac{2}{2-u}\,du=0$$

Nakon integraljenja i zamjene $u$ sa $x+y$ dobijamo

$$2x-3u-2\ln(2-u)=C_1$$

$$2x-3(x+y)-2\ln(2-x-y)=C_1$$

$$x+3y+2\ln(2-x-y)=C$$ traženo opšte rješenje

# Linearna diferencijalna jednačina

su oblika $y'+p(x)\cdot y = g(x)$. uvodimo smjenu $y=uv$.

① Riješiti diferencijalnu jednačinu $(1+x^2)y' = x(2y+1)$.

Rj. $(1+x^2)y' = 2xy + x$

$(1+x^2)y' - 2xy = x \quad /:(1+x^2)$

$y' - \frac{2x}{1+x^2}y = \frac{x}{1+x^2}$ ovo je lin. dif. jedn.

$y=u\cdot v$, $y'=u'\cdot v+u\cdot v'$

uvrstimo smjenu

$u'\cdot v+u\cdot v' - \frac{2x}{1+x^2}uv = \frac{x}{1+x^2}$

$u'\cdot v + u\cdot \underbrace{\left(v' - \frac{2x}{1+x^2}v\right)} = \frac{x}{1+x^2}$

ovaj dio izjednačimo sa 0 da bi našli v

a) $v' - \frac{2x}{1+x^2}v = 0$, $v'=\frac{dv}{dx}$

$\frac{dv}{dx} = \frac{2x}{1+x^2}v$

$\frac{dv}{v} = \frac{2x}{1+x^2}dx \quad /\int$

$\int\frac{dv}{v} = 2\int\frac{x}{1+x^2}dx \quad \left|\begin{matrix}1+x^2=t\\ 2xdx=dt\end{matrix}\right|$

$\ln|v| = \ln|1+x^2|$

$v=1+x^2$

b) $u'v = \frac{x}{1+x^2}$

$u'(1+x^2) = \frac{x}{1+x^2}$, $u'=\frac{du}{dx}$

$\frac{du}{dx} = \frac{x}{(1+x^2)^2}$, $du = \frac{x}{(1+x^2)^2}dx \quad /\int$

$\int du = \int\frac{x}{(1+x^2)^2}dx \quad \left|\begin{matrix}1+x^2=t\\ 2xdx=dt\\ xdx=\frac{1}{2}dt\end{matrix}\right|$

$u = -\frac{1}{2(1+x^2)} + c$

$y = u\cdot v = \left[-\frac{1}{2(1+x^2)}+c\right](1+x^2)$

$y = c(1+x^2) - \frac{1}{2}$ opšte rješenje diferencijalne jednačine

②✓ Riješiti diferencijalnu jednačinu $xy' - \frac{y}{x+1} = x$ ako je $y(1)=-1$.

Rj. $y=\frac{x}{x+1}(x+\ln|x|+c)$ opšte rješenje diferenc. jedn.

$y=\frac{x}{x+1}(x+\ln|x|-3)$ partikularno rješenje diferenc. jedn.

③✓ Riješiti diferencijalnu jednačinu

$y'+y\cos x = 0,5\sin 2x$

Rj. $y=1-\sin x+ce^{-\sin x}$ opšte rješenje dif. jedn.

(#) Riješiti diferencijalnu jednačinu

$$y' - \frac{xy}{1+x^2} = \frac{x\sqrt{1+x^2}}{x^2-2x+2}.$$

Rj. $y' - \frac{x}{1+x^2}y = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$ ovo je linearna diferencijalna jednačina uvodimo smjenu $y=uv$

$y=uv,\ y'=u'v+uv'$

$$u'v+uv' - \frac{x}{1+x^2}uv = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$$

$$u'v + u\underbrace{\left(v' - \frac{x}{1+x^2}v\right)}_{=0} = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$$

a)

$$v' - \frac{x}{1+x^2}v = 0$$

$$\frac{dv}{dx} = \frac{x}{1+x^2}v \qquad /:v$$

$$\frac{dv}{v} = \frac{x}{1+x^2}dx \qquad //\int$$

$$\int\frac{dv}{v} = \int\frac{x}{1+x^2}dx$$

$$\int\frac{x}{1+x^2}dx = \left|\begin{array}{l} 1+x^2=t \\ 2x\,dx=dt \\ x\,dx=\frac{1}{2}dt \end{array}\right| = \frac{1}{2}\int\frac{dt}{t}$$

$$= \frac{1}{2}\ln|t| + c = \ln|1+x^2|^{\frac{1}{2}} + c$$

$$\ln|v| = \ln\sqrt{1+x^2}.$$

$$v = \sqrt{1+x^2}$$

b)

$$u'v = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$$

$$\frac{du}{dx}\sqrt{1+x^2} = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$$

$$du = \frac{x}{x^2-2x+2}dx \qquad //\int$$

$$\int\frac{\overset{-1+1}{x}}{x^2-2x+2}dx = \int\frac{x-1}{x^2-2x+2}dx + \int\frac{dx}{x^2-2x+2}$$

$$= \left|\begin{array}{ll} x^2-2x+2=t & x^2-2x+2= \\ (2x-2)dx=dt & x^2-2x+1+1= \\ (x-1)dx=\frac{1}{2}dt & =(x-1)^2+1 \end{array}\right|$$

$$= \frac{1}{2}\int\frac{dt}{t} + \int\frac{dx}{(x-1)^2+1} =$$

$$= \frac{1}{2}\ln|t| + \operatorname{arc\,tg}(x-1) + c$$

$$= \frac{1}{2}\ln|x^2-2x+2| + \operatorname{arc\,tg}(x-1) + c$$

$$u = \frac{1}{2}\ln|x^2-2x+2| + \operatorname{arc\,tg}(x-1) + c$$

$$y=uv = \left(\frac{1}{2}\ln|x^2-2x+2| + \operatorname{arc\,tg}(x-1) + c\right)\sqrt{1+x^2} =$$

$$= c\sqrt{1+x^2} + \sqrt{1+x^2}\left(\frac{1}{2}\ln|x^2-2x+2| + \operatorname{arc\,tg}(x-1)\right)$$

rješenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu $(x^2+2x-2y)dx - dy = 0$.

Rj.

$$(x^2+2x-2y)\,dx - dy = 0 \quad /: dx$$

$$x^2+2x-2y-y' = 0$$

$$y' + 2y = x^2+2x$$ Ovo je linearna diferencijalna jednačina

Uvodimo smjenu $y = uv \quad / \frac{d}{dx}$

$$y' = u'v + uv'$$

$$u'v + uv' + 2uv = x^2+2x$$

$$u'v + u\underbrace{(v'+2v)}_{=0} = x^2+2x$$

a) $v'+2v = 0$

$$\frac{dv}{dx} = -2v$$

$$\frac{dv}{v} = -2\,dx \quad /\int$$

$$\ln v = -2x$$

$$v = e^{-2x}$$

b) $u'v + u\cdot 0 = x^2+2x$

$$u'v = x^2+2x$$

$$u'e^{-2x} = x^2+2x$$

$$\frac{du}{dx} = \frac{x^2+2x}{e^{-2x}}$$

$$du = \frac{x^2+2x}{e^{-2x}}\,dx$$

$$du = (x^2+2x)e^{2x}\,dx \quad \ldots(*)$$

$2x=t$
$2dx=dt$
$dx=\frac{1}{2}dt$

$$\int (x^2+2x)e^{2x}dx = \begin{vmatrix} u=x^2+2x & dv=e^{2x}dx \\ du=2x+2 & v=\frac{1}{2}e^{2x} \end{vmatrix} = \frac{1}{2}e^{2x}(x^2+2x) - \int (x+1)e^{2x}dx$$

$$\int (x+1)e^{2x}dx = \begin{vmatrix} u=x+1 & dv=e^{2x}dx \\ du=dx & v=\frac{1}{2}e^{2x} \end{vmatrix} = \frac{1}{2}(x+1)e^{2x} - \frac{1}{2}\int e^{2x}dx$$

$$\int (x^2+2x)e^{2x}dx = \frac{1}{2}e^{2x}(x^2+2x) - \frac{1}{2}e^{2x}(x+1) + \frac{1}{4}e^{2x} + C$$

$$= \frac{1}{2}x^2e^{2x} + \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

$$(*) \Rightarrow \quad u = \frac{1}{2}x^2e^{2x} + \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

$$y = uv = \left(\frac{1}{2}x^2e^{2x} + \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C\right)e^{-2x} =$$

$$= \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{4} + Ce^{-2x}$$ opšte rješenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu $y'\cos x - y\sin x = x^3 e^{x^2}$ uz početni uslov $y(0)=1$.

Rj. $y'\cos x - y\sin x = x^3 e^{x^2} \quad /:\cos x$

$$y' - y\,tg x = \frac{x^3 e^{x^2}}{\cos x}$$

ovo je linearna diferencijalna jednačina

uvodimo smjenu

$y = uv$

$y' = u'v + uv'$

$$u'v + uv' - uv\,tg x = \frac{x^2 e^{x^2}}{\cos x}$$

$$u'v + \underbrace{u(v' - v\,tg x)}_{=0} = \frac{x^3 e^{x^2}}{\cos x}$$

$$\int tg x\,dx = \left| \begin{matrix} tg x = t \\ x = arctg\, t \\ dx = \frac{dt}{1+t^2} \end{matrix} \right| = \int \frac{t}{1+t^2}\,dt = \left| \begin{matrix} 1+t^2 = s \\ 2t\,dt = ds \\ t\,dt = \frac{ds}{2} \end{matrix} \right| = \frac{1}{2}\int \frac{ds}{s} =$$

$$= \frac{1}{2}\ln|s| = \frac{1}{2}\ln|1+t^2| =$$

$$= \frac{1}{2}\ln|1+tg^2 x| = \frac{1}{2}\ln\left|1+\frac{\sin^2 x}{\cos^2 x}\right|$$

$$= \frac{1}{2}\ln\left|\frac{\cos^2 x + \sin^2 x}{\cos^2 x}\right| = \ln\left|\frac{1}{\cos x}\right|$$

$v' - v\,tg x = 0$

$\frac{dv}{dx} = v\,tg x$

$\frac{dv}{v} = tg x\,dx$

$\int \frac{dv}{v} = \int tg x\,dx$

$\ln v = \ln\left|\frac{1}{\cos x}\right|$

$v = \frac{1}{\cos x}$

$u'v = \frac{x^3 e^{x^2}}{\cos x}$

$u' \cdot \frac{1}{\cos x} = \frac{x^3 e^{x^2}}{\cos x} \quad /\cdot\cos x$

$\frac{du}{dx} = x^3 e^{x^2}$

$du = x^3 e^{x^2}\,dx$

$$I = \int x^3 e^{x^2}\,dx = \left| \begin{matrix} u = x^2 & dv = xe^{x^2}dx \\ du = 2x & v = \int xe^{x^2}dx = \left|\begin{matrix} x^2 = t \\ 2x\,dx = dt \\ x\,dx = \frac{dt}{2}\end{matrix}\right| = \frac{1}{2}\int e^t dt = \frac{1}{2}e^{x^2} \end{matrix} \right| =$$

$$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}\cdot 2\int x\,e^{x^2} = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + c$$

$du = x^3 e^{x^2}\,dx \quad /\int$

$u = \frac{1}{2}e^{x^2}(x^2-1) + c_1$

$$y = uv = \frac{e^{x^2}(x^2-1)+c}{2\cos x}$$

opće rješenje diferencijalne jednačine

$y(0) = 1$

$$y(0) = \frac{e^0(0-1)+c}{2\cos 0} = \frac{-1+c}{2} = 1$$

$-1 + c = 2$

$c = 3$

$$y = \frac{e^{x^2}(x^2-1)+3}{2\cos x}$$

partikularno rješenje diferencijalne jednačine

(#) Odrediti rješenje diferencijalne jednačine

$xy' - \frac{y}{x+1} = x$ koje zadovoljava uslov $y(1)=0$.

Rj.

$$xy' - \frac{y}{x+1} = x \quad /:x$$

$$y' - \frac{1}{x(x+1)}y = 1$$

ovo je linearna diferencijalna jednačina $y' + p(x)y = \varphi(x)$, $\left(p(x) = -\frac{1}{x(x+1)}, \varphi(x) = 1\right)$

uvodimo smjene $y=uv$, $y'=u'v+uv'$, gdje su $u$ i $v$ pomoćne f-je koje treba odrediti

$$u'v + uv' - \frac{1}{x(x+1)}uv = 1$$

$$u'v + u\underbrace{\left(v' - \frac{1}{x(x+1)}v\right)} = 1$$

a) $v' - \frac{1}{x(x+1)}v = 0$

$v' = \frac{dv}{dx}$, $\frac{dv}{dx} = \frac{v}{x(x+1)}$

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \quad /\cdot x(x+1)$$

$$1 = A(x+1) + Bx$$

$$A + B = 0$$

$$A = 1 \Rightarrow B = -1$$

$$\frac{dv}{v} = \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$\ln|v| = \ln|x| - \ln|x+1|$$

$$\ln|v| = \ln\frac{x}{x+1}$$

$$v = \frac{x}{x+1}$$

b) $u'v + u\underbrace{\left(v' - \frac{1}{x(x+1)}v\right)} = 1$

za $v = \frac{x}{x+1}$ ovaj dio je jednak nuli

$u'\frac{x}{x+1} = 1$, $u' = \frac{du}{dx}$

$$\frac{du}{dx} = \frac{x+1}{x}$$

$$du = \left(1 + \frac{1}{x}\right)dx$$

$$u = x + \ln|x| + C$$

$$y = \frac{x}{x+1}(x + \ln|x| + C)$$

je opšte rješenje diferencijalne jednačine

$y(1)=0 \Rightarrow x=1, y=0$

$$0 = \frac{1}{2}(1 + \underbrace{\ln 1}_{=0} + c) \Rightarrow 1 + C = 0$$

$$C = -1$$

$y = \frac{x}{x+1}(x - 1 + \ln|x|)$ je traženo rješenje (partikularno rješenje diferencijalne jednačine)

(#) Odrediti rješenje diferencijalne jednačine $xy' + y - e^x = 0$ koje zadovoljava uslov $y(a) = b$.

Rj:

$$xy' + y = e^x \quad /:x$$

$$y' + \frac{1}{x}y = \frac{1}{x}e^x$$

ovo je linearna diferencijalna jednačina $y' + p(x)y = q(x)$, $\left(p(x) = \frac{1}{x},\ q(x) = \frac{1}{x}e^x\right)$, uvodimo smjene $y = uv$, $y' = u'v + uv'$, gdje su $u$ i $v$ pomoćne f-je koje treba odrediti

$$u'v + uv' + \frac{1}{x}uv = \frac{1}{x}e^x$$

$$u'v + \underbrace{u\left(v' + \frac{v}{x}\right)} = \frac{e^x}{x}$$

a) $v' + \frac{v}{x} = 0$

$v' = \frac{dv}{dx}$ $\qquad \frac{dv}{dx} = -\frac{v}{x}$

$$\frac{dv}{v} = -\frac{dx}{x}$$

$$\ln|v| = -\ln|x|$$

$$v = x^{-1} = \frac{1}{x}$$

b) $u'v + \underbrace{u\left(v' + \frac{v}{x}\right)} = \frac{e^x}{x}$

za $v = \frac{1}{x}$ ovaj dio je jednak nuli

$$u' \cdot \frac{1}{x} = \frac{e^x}{x} \quad /\cdot x$$

$u' = \frac{du}{dx}$ $\qquad \frac{du}{dx} = e^x$

$$du = e^x dx$$

$$u = e^x + c$$

$$y = \frac{e^x + c}{x}$$ opšte rješenje diferencijalne jednačine

$y(a) = b$

$x = a,\ y = b$

$$b = \frac{e^a + c}{a} \Rightarrow e^a + c = ab$$

$$c = ab - e^a$$

$$y = \frac{e^x + ab - e^a}{x}$$ je traženo rješenje (partikularno rješenje diferencijalne jednačine)

(#) Odrediti rješenje diferencijalne jednačine $y' - y\,\mathrm{tg}\,x = \frac{1}{\cos x}$ koje zadovoljava uslov $y(0)=0$.

Rj.

$y' - y\,\mathrm{tg}\,x = \frac{1}{\cos x}$ je linearna diferencijalna jednačina $y' + p(x)y = g(x)$, uvodimo smjenu $y=uv$, $y'=u'v+uv'$ gdje su $u$ i $v$ pomoćne f-je koje treba odrediti.

$$u'v + uv' - uv\,\mathrm{tg}\,x = \frac{1}{\cos x}$$

$$u'v + u\underbrace{(v' - v\,\mathrm{tg}\,x)} = \frac{1}{\cos x}$$

a) $v' - v\,\mathrm{tg}\,x = 0$

$$v' = \frac{dv}{dx}$$

$$\frac{dv}{dx} = v\,\mathrm{tg}\,x$$

$$\frac{dv}{v} = \frac{\sin x}{\cos x}\,dx$$

$$\frac{dv}{v} = \frac{-d(\cos x)}{\cos x} \quad /\int$$

$$\ln|v| = -\ln|\cos x|$$

$$v = (\cos x)^{-1}$$

$$v = \frac{1}{\cos x}$$

b) $u'v + u\underbrace{(v' - v\,\mathrm{tg}\,x)} = \frac{1}{\cos x}$

ovaj dio je jednak nuli za $v = \frac{1}{\cos x}$

$$u'\frac{1}{\cos x} = \frac{1}{\cos x} \quad /\cdot\cos x$$

$$u' = \frac{du}{dx}$$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$u = x + c$$

$y = \frac{x+c}{\cos x}$ je opšte rješenje diferencijalne jednačine

$$y(0)=0$$

$$x=0,\ y=0$$

$$0 = \frac{0+c}{\underbrace{\cos 0}_{=1}} \Rightarrow c=0$$

$y = \frac{x}{\cos x}$ je traženo rješenje

(partikularno rješenje diferencijalne jednačine)

(#) Odrediti rješenje diferencijalne jednačine $xy' - 3y = x^4 e^x$ tako da je $y(1) = e$.

Rj.

$$xy' - 3y = x^4 e^x \quad /:x$$

$$y' - 3\frac{1}{x}y = x^3 e^x$$

ovo je linearna diferencijalna jednačina $y' + p(x)y = q(x)$

uvodimo smjenu $y = uv$

$y' = u'v + uv'$

gdje su u i v pomoćne f-je koje treba odrediti

$$u'v + uv' - \frac{3}{x}uv = x^3 e^x$$

$$u'v + u\underbrace{\left(v' - \frac{3}{x}v\right)} = x^3 e^x$$

a) $v' - \frac{3}{x}v = 0$

$$v' = \frac{dv}{dx}$$

$$\frac{dv}{dx} = 3\frac{v}{x}$$

$$\frac{dv}{v} = 3\frac{dx}{x}$$

$$\ln|v| = 3\ln|x|$$

$$v = x^3$$

b) $u'v + u\underbrace{\left(v' - \frac{3}{x}v\right)} = x^3 e^x$

za $v = x^3$ ovaj dio je jednako

$$u'x^3 = x^3 e^x$$

$$u' = \frac{du}{dx}$$

$$\frac{du}{dx} = e^x$$

$$du = e^x dx \qquad u = e^x + c$$

$y = x^3(e^x + c)$ je opšte rješenje diferencijalne jednačine

$$y(1) = e \Rightarrow 1^3(e^1 + c) = e$$

$$e + c = e$$

$$c = 0$$

$y = x^3 e^x$ je traženo rješenje (partikularno rješenje diferencijalne jednačine)

(#) Rješiti diferencijalnu jednačinu

$$y' - y\,ctg\,x = \sin x$$

Rj. Primjetimo da imamo linearnu diferencijalnu jednačinu.

Uvodimo smjenu $y = uv$

$y' = u'v + uv'$ gdje su u i v dvije pomoćne f-je

(želimo dobiti dvije diferencijalne jednačine sa razdvojenim promjenjivim)

$$u'v + uv' - uv\,ctg\,x = \sin x$$

$$u'v + \underbrace{u(v' - v\,ctg\,x)}_{=0} = \sin x$$

(a)

$$v' - v\,ctg\,x = 0$$

$$\frac{dv}{dx} = v\,ctg\,x$$

$$\frac{dv}{v} = ctg\,x\,dx$$

$$\frac{dv}{v} = \frac{\cos x}{\sin x}\,dx$$

$$\ln v = \ln \sin x$$

$$v = \sin x$$

(b)

$$u'v = \sin x$$

$$u' \sin x = \sin x$$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$u = x + C$$

Opšte rješenje diferencijalne jednačine je $y = (x + C)\sin x$.

(#) Rješiti diferencijalnu jednačinu

$$y' + y\,tg\,x = \cos x$$

Rj. Data jednačina je linearna dif. jedn. Uvodimo smjenu $y = uv$

$y' = u'v + uv'$ (u i v su dvije pomoćne f-je

cilj je dobiti dvije dif. jedn. sa razdvojenim promjenjivim).

$$u'v + uv' + uv\,tgx = \cos x$$

$$u'v + \underbrace{u(v' + v\,tgx)}_{=0} = \cos x$$

(a)

$$v' + v\,tgx = 0$$

$$\frac{dv}{dx} = -v\,tgx$$

$$\frac{dv}{v} = \frac{-\sin x}{\cos x}dx$$

$$\frac{dv}{v} = \frac{d(\cos x)}{\cos x} \quad \int$$

$$\ln|v| = \ln|\cos x|$$

$$v = \cos x$$

(b)

$$u'v = \cos x$$

$$\frac{du}{dx}\cos x = \cos x$$

$$du = dx$$

$$u = x + C$$

Opšte rješenje diferencijalne jednačine je

$$y = (x + C)\cos x$$

(#) Rješiti diferencijalnu jednačinu

$$x^2y^2y'+xy^3=y^2$$

Rj.

$x^2y^2y'+xy^3=y^2 \quad /:y^2$

$x^2y'+xy=1 \quad /:x^2$

$y'+\frac{1}{x}y=\frac{1}{x^2}$

ovo je linearna diferencijalna jednačina (uvodimo smjenu $y=uv$ gdje su $u$ i $v$ dvije pomoćne f-je – cilj je dobiti dvije dif. jedn. sa razdvojenim promjenjivim)

$u'v+uv'+\frac{1}{x}uv=\frac{1}{x^2}$

$u'v+u\underbrace{(v'+\frac{1}{x}v)}_{=0}=\frac{1}{x^2}$

(a)

$v'+\frac{v}{x}=0$

$\frac{dv}{dx}=\frac{-v}{x}$

$\frac{dv}{v}=-\frac{dx}{x} \quad /\int$

$\ln|v|=-\ln|x|$

$v=\frac{1}{x}$

(b)

$u'v=\frac{1}{x^2}$

$u'\cdot\frac{1}{x}=\frac{1}{x^2} \quad /\cdot x$

$\frac{du}{dx}=\frac{1}{x}$

$u=\ln|x|+c$

Opšte rješenje diferencijalne jednačine je

$$y=(\ln x+c)\cdot\frac{1}{x}$$

⑦ Rješiti diferencijalnu jednačinu

$$y' - y\sin 2x = e^{\sin^2 x}$$

Rj. Primjetimo da imamo linearnu diferencijalnu jednačinu.

Uvodimo smjenu $y = uv$

$y' = u'v + uv'$

$$u'v + uv' - uv\sin 2x = e^{\sin^2 x}$$

$$u'v + u\underbrace{(v' - v\sin 2x)}_{=0} = e^{\sin^2 x}$$

(a)

$$v' - v\sin 2x = 0$$

$$v' = v \cdot 2\sin x\cos x$$

$$\frac{dv}{v} = 2\sin x\cos x\, dx$$

$$\frac{dv}{v} = 2\sin x\, d(\sin x) \quad \Big/\!\!\int$$

$$\ln v = 2 \cdot \frac{1}{2}\sin^2 x$$

$$v = e^{\sin^2 x}$$

(b)

$$u'v = e^{\sin^2 x}$$

$$u' e^{\sin^2 x} = e^{\sin^2 x}$$

$$u' = 1$$

$$\frac{du}{dx} = 1 \Rightarrow du = dx$$

$$u = x + C$$

Opšte rješenje diferencijalne jednačine je

$$y = (x+c)\, e^{\sin^2 x}$$

(#) Rješiti diferencijalnu jednačinu

$$(x-2)\frac{dy}{dx} = y + 2(x-2)^3 .$$

Rj:

$$(x-2)\,y' - y = 2(x-2)^3 \qquad /:(x-2)$$

$$y' - \frac{1}{x-2}\,y = 2(x-2)^2$$

ovo je linearna diferencijalna jednačina

$$y = uv$$
$$y' = u'v + uv'$$

$$u'v + uv' - \frac{1}{x-2}\,uv = 2(x-2)^2$$

$$u'v + u\left(v' - \frac{v}{x-2}\right) = 2(x-2)^2$$

(a)

$$v' - \frac{v}{x-2} = 0$$

$$\frac{dv}{dx} = \frac{v}{x-2}$$

$$\frac{dv}{v} = \frac{dx}{x-2}$$

$$\ln v = \ln(x-2)$$

$$v = x-2$$

(b)

$$u' \cdot (x-2) = 2(x-2)^2 \qquad /:(x-2)$$

$$\frac{du}{dx} = 2(x-2)$$

$$du = \underbrace{2(x-2)\,dx}_{2(x-2)\,d(x-2)}$$

$$u = (x-2)^2 + C$$

Opšte rješenje diferencijalne jednačine je

$$y = (x-2)^3 + C(x-2)$$

㊀ Riješiti diferencijalnu jednačinu $\frac{dy}{dx}+2xy=4x$

Rj:

Prisjetimo se jednog od načina rješavanja Jednačina oblika

$$\frac{dy}{dx}+yP(x)=Q(x), \qquad \ldots(1)$$

u kojoj je lijeva strana linearna i po zavisnoj varijabli i po izvodu se naziva linearna jednačina prvog reda. Na primjer $\frac{dy}{dx}+3xy=\sin x$ je linearna jednačina, dok npr. $\frac{dy}{dx}+3xy^2=\sin x$ nije.

Kako je

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right)=\frac{dy}{dx}e^{\int P(x)dx}+yP(x)e^{\int P(x)dx}=e^{\int P(x)dx}\left(\frac{dy}{dx}+yP(x)\right)$$

imamo da je $e^{\int P(x)dx}$ integrativni faktor, pa opšte rješenje od (1) dobijamo iz

$$ye^{\int P(x)dx}=\int Q(x)\cdot e^{\int P(x)dx}dx+C$$

I način:

$\int P(x)dx=\int 2x\,dx=x^2$, pa je $e^{\int P(x)dx}=e^{x^2}$ integrativni faktor

Tada

$$ye^{x^2}=\int 4xe^{x^2}dx=\left|\begin{matrix} d(x^2)=2x\,dx \\ 4x\,dx=2\,d(x^2)\end{matrix}\right|=2\int e^{x^2}d(x^2)=2e^{x^2}+C$$

Opšte rješenje diferencijalne jednačine je $y=2+Ce^{-x^2}$.

II način

$y' + 2xy = 4x$

uvodimo smjenu $y = uv$

$y' = u'v + uv'$ (u i v su dvije pomoćne f-je koje trebamo odrediti i koje zavise od x)

$u'v + uv' + 2xuv = 4x$

$u'v + u(v' + 2xv) = 4x$

(a)

$v' + 2xv = 0$

$\frac{dv}{dx} = -2xv$

$\frac{dv}{v} = -2x\,dx \quad /\int$

$\ln v = -2 \cdot \frac{x^2}{2}$

$\ln v = -x^2$

$v = e^{-x^2}$

(b)

$u'v = 4x$

$\frac{du}{dx} e^{-x^2} = 4x$

$du = 4xe^{x^2} dx \quad /\int$

$u = 2e^{x^2} + c$

$y = uv$

$y = 2 + Ce^{-x^2}$

traženo opšte rješenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu $x\frac{dy}{dx} = y + x^3 + 3x^2 - 2x$.

Rj.

I način

$$\boxed{\begin{gathered}\frac{dy}{dx} + yP(x) = Q(x) \\ y e^{\int P(x)\,dx} = \int Q(x) e^{\int P(x)\,dx} dx + C\end{gathered}}$$

Naša jednačina je

$$\frac{dy}{dx} - \frac{1}{x}y = x^2 + 3x - 2$$

$\int P(x)\,dx = -\int \frac{dx}{x} = -\ln x$, pa je

$e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$ integrativni faktor.

Tada je

$$y\frac{1}{x} = \int (x^2+3x-2)\frac{1}{x}\,dx = \int \left(x+3-\frac{2}{x}\right) dx = \frac{1}{2}x^2 + 3x - 2\ln x + C_1$$

$$2y = x^3 + 6x^2 - 4x\ln x + Cx$$

opšte rješenje diferencijalne jednač

II način

$$y' - \frac{1}{x}y = x^2 + 3x - 2$$

uvodimo smjenu $y = uv$

$y' = u'v + uv'$ (gdje su $u$ i $v$ dvije pomoćne f-je promjenjive $x$, koje trebamo odrediti.

$$u'v + uv' - \frac{1}{x}uv = x^2 + 3x - 2$$

$$u'v + u\left(v' - \frac{1}{x}v\right) = x^2 + 3x - 2$$

(a) $v' - \frac{1}{x} v = 0$

$$\frac{dv}{dx} = \frac{v}{x}$$

$$\frac{dv}{v} = \frac{dx}{x}$$

$$\ln v = \ln x$$

$$v = x$$

(b) $u'v = x^2 + 3x - 2$

$$\frac{du}{dx} \cdot x = x^2 + 3x - 2$$

$$du = \left(x + 3 - \frac{2}{x}\right) dx$$

$$u = \frac{1}{2} x^2 + 3x - 2 \ln x + C_1$$

$$y = uv$$

$$y = \frac{1}{2} x^3 + 3x^2 - 2x \ln x + x C_1 \quad / \cdot 2$$

$$2y = x^3 + 6x^2 - 4x \ln x + x C$$

opće rješenje
diferencijalne jednačine

# Bernulijeva diferencijalna jednačina

su oblika $y' + p(x)y = g(x)y^n$, $n \in \mathbb{R}$, $n \neq 0$ i $n \neq 1$

uvodimo smjenu $y = uv$ ove dif. jedn. rješavamo na isti način kao što smo rješavali linearne dif. jed

① Riješiti diferencijalnu jednačinu $xy' - x^2\sqrt{y} = 4y$.

Rj. $xy' - 4y = x^2\sqrt{y} \quad /: x$

$y' - \frac{4}{x}y = x\sqrt{y}$ ovo je Bern. dif. jedn.

smjena $y = uv$

$y' = u'v + uv'$

$u'v + uv' - \frac{4}{x}uv = x\sqrt{uv}$

$u'v + u\underbrace{(v' - v\frac{4}{x})}_{=0} = x\sqrt{uv}$

(da bi smo našli v)

a) $v' - v\frac{4}{x} = 0 \Rightarrow v' = v\frac{4}{x}$

$v' = \frac{dv}{dx}$, $\frac{dv}{v} = \frac{4}{x}dx \quad //\int$

$\int\frac{dv}{v} = \int\frac{4}{x}dx \Rightarrow \ln|v| = 4\ln|x|$

$v = x^4$

b) $u'v = x\sqrt{uv}$

$u'x^4 = x\sqrt{ux^4}$, $x^4u' = x^3\sqrt{u}$

$\frac{du}{dx} = x^{-1}\sqrt{u}$, $\frac{du}{\sqrt{u}} = \frac{dx}{x} \quad //\int$

$\int\frac{du}{\sqrt{u}} = \int\frac{dx}{x} \Rightarrow 2\sqrt{u} = \ln x + c$

$\sqrt{u} = \frac{\ln x + c}{2}$

$u = \frac{(\ln x + c)^2}{4}$

$y = uv = \frac{(\ln x + c)^2}{4} \cdot x^4$

$y = \frac{x^4}{4}(\ln x + c)^2$ opšte rješenje difer. jedn.

② ✓ Naći partikularno rješenje diferencijalne jednačine $y' = xy^3 - y$ koje prolazi kroz tačku $A(0,1)$.

Rj. $y^{-2} = e^{2x}[e^{-2x}(x + \frac{1}{2}) + c]$ opšte rješenje dif. jedn.

$y^{-2} = \frac{1}{2}e^{2x} + x + \frac{1}{2}$ partikul. rješ. dif. jed

③ ✓ Riješiti diferencijalnu jednačinu

$(1-x^2)y' = xy + xy^2$ Rj. $y = \frac{c}{\sqrt{1-x^2}} - 1$

(#) Riješiti diferencijalnu jednačinu

$$y' + \frac{y}{4x} + y^3 e^{\sqrt{x}} = 0 \quad \text{ako je} \quad y(1) = 1.$$

Rj.

$$y' + \frac{1}{4x} y = -e^{\sqrt{x}} y^3$$

ovo je Bernulijeva diferencijalna jednačina.

uvodimo smjenu $y = uv$

$$y' = u'v + uv'$$

$$u'v + uv' + \frac{1}{4x} uv = -e^{\sqrt{x}} u^3 v^3$$

$$u'v + u\underbrace{\left(v' + \frac{1}{4x} v\right)}_{=0} = -e^{\sqrt{x}} u^3 v^3$$

a) $v' + \frac{1}{4x} v = 0$

$$\frac{dv}{dx} = \frac{-v}{4x}$$

$$\frac{dv}{v} = \frac{-dx}{4x}$$

$$\frac{dv}{v} = -\frac{1}{4} \cdot \frac{dx}{x} \quad \Big/ \int$$

$$\ln v = -\frac{1}{4} \ln|x|$$

$$\ln v = \ln|x|^{-\frac{1}{4}}$$

$$v = \frac{1}{\sqrt[4]{x}}$$

b) $u'v = -e^{\sqrt{x}} u^3 v^3$

$$u' \cdot \frac{1}{\sqrt[4]{x}} = -e^{\sqrt{x}} u^3 \frac{1}{\sqrt[4]{x^3}} \quad \Big/ \cdot \sqrt[4]{x}$$

$$\frac{du}{dx} = -e^{\sqrt{x}} \frac{u^3}{\sqrt[4]{x^2}}$$

$$\frac{du}{u^3} = -\frac{e^{\sqrt{x}}}{\sqrt[4]{x^2}} dx$$

$$\frac{du}{u^3} = -\frac{e^{\sqrt{x}}}{\sqrt{x}} dx \quad \Big/ \int$$

$$\frac{u^{-2}}{-2} = -2e^{\sqrt{x}} + c_1 \quad \Big/ \cdot (-2)$$

$$\frac{1}{u^2} = 4e^{\sqrt{x}} + c.$$

$$(e^{\sqrt{x}})' = e^{\sqrt{x}} \cdot (\sqrt{x})' = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$$y = uv = \frac{1}{\sqrt[4]{x}\sqrt{4e^{\sqrt{x}} + c}}$$

opšte rješenje diferenc. jedn.

$$u^2 = \frac{1}{4e^{\sqrt{x}} + c} \Rightarrow u = \frac{1}{\sqrt{4e^{\sqrt{x}} + c}}$$

$$y(1) = 1 \Rightarrow \frac{1}{\sqrt{4e + c}} = 1$$

$$\sqrt{4e + c} = 1$$

$$4e + c = 1 \Rightarrow c = 1 - 4e$$

$$y = \frac{1}{\sqrt[4]{x}\sqrt{4e^{\sqrt{x}} + 1 - 4e}}$$

partikularno rješenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu $y' = y^4 \cos x + y\, tg x$,

Rj: $y' - y\, tg x = \cos x\, y^4$ ovo je Bernulijeva diferencijalna jednačina, uvodimo smjenu $y = uv$, $y' = u'v + uv'$.

$$u'v + uv' - uv\, tg x = u^4 v^4 \cos x$$

$$u'v + u \underbrace{(v' - v\, tg x)}_{=0} = u^4 v^4 \cos x$$

$$\int tg x\, dx = \left| \begin{array}{l} tg x = t \\ x = arctg\, t \\ dx = \frac{dt}{1+t^2} \end{array} \right| = \int \frac{t}{1+t^2} dt =$$

$$= \left| \begin{array}{l} 1+t^2 = s \\ 2t dt = ds \\ t dt = \frac{ds}{2} \end{array} \right| = \frac{1}{2} \int \frac{ds}{s} = \frac{1}{2} \ln|s| + C = \frac{1}{2} \ln|1+t^2| + C$$

$$= \frac{1}{2} \ln\left|1 + \frac{\sin^2 x}{\cos^2 x}\right| + C = \frac{1}{2} \ln\left|\frac{1}{\cos^2 x}\right| + C = \ln \frac{1}{\cos x} + C$$

$$v' - v\, tg x = 0$$

$$v' = v\, tg x$$

$$\frac{dv}{dx} = v\, tg x$$

$$\frac{dv}{v} = tg x\, dx \quad /\int$$

$$\ln v = \ln \frac{1}{\cos x}$$

$$v = \frac{1}{\cos x}$$

$$u'v = u^4 v^4 \cos x \quad /:v$$

$$u' = u^4 v^3 \cos x$$

$$u' = u^4 \cdot \frac{1}{\cos^2 x}$$

$$\frac{u'}{u^4} = \frac{1}{\cos^2 x}, \quad u' = \frac{du}{dx}$$

$$\frac{du}{u^4} = \frac{dx}{\cos^2 x} \quad /\int$$

$$\int \frac{du}{u^4} = \int \frac{dx}{\cos^2 x}$$

$$\int u^{-4} du = \int \frac{dx}{\cos^2 x}$$

$$\frac{u^{-3}}{-3} = tg x + C_1$$

$$\frac{1}{u^3} = -3 tg x + C.$$

$$\frac{1}{v} = \cos x$$

$$\frac{1}{u^3} = -3 \cdot \frac{\sin x}{\cos x} + C$$

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$$\frac{1}{v^3} = \cos^3 x$$

$$\frac{1}{u^3} = -3 \frac{\sin x}{\cos x} + C$$

$$\frac{1}{y^3} = \frac{1}{u^3 v^3} = -3 \frac{\sin x}{\cos x} \cdot \cos^3 x + C \cdot \cos^3 x$$

$$y^{-3} = -3 \sin x \cos^2 x + C \cos^3 x$$

rješenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu $y' = \frac{3x^2}{x^3+y+1}$.

Rj.

$$y' = \frac{3x^2}{x^3+y+1}$$

$$\frac{dy}{dx} = \frac{3x^2}{x^3+y+1}$$

$$\frac{dx}{dy} = \frac{x^3+y+1}{3x^2}$$

$$x' = \frac{1}{3}x + \frac{1}{3}yx^{-2} + \frac{1}{3}x^{-2}$$

$$x' - \frac{1}{3}x = \left(\frac{1}{3}y + \frac{1}{3}\right)x^{-2}$$

ovo je Bernulijeva diferencijalna jednačina

Bernulijeva diferencijalna jednačina je oblika $y' + p(x)y = q(x)\cdot y^n$

$n \in \mathbb{R}$,

$n \neq 0$, $n \neq 1$

Uvodimo smjenu

$$x = uv, \quad x' = u'v + uv'$$

$$u'v + uv' - \frac{1}{3}uv = \left(\frac{1}{3}y + \frac{1}{3}\right)(uv)^{-2}$$

$$u'v + u\left(v' - \frac{1}{3}v\right) = \left(\frac{1}{3}y + \frac{1}{3}\right)u^{-2}v^{-2}$$

a) $v' - \frac{1}{3}v = 0$

$v' = \frac{1}{3}v$

$$\frac{dv}{dy} = \frac{1}{3}v$$

$$\frac{dv}{v} = \frac{1}{3}dy$$

$$\ln v = \frac{1}{3}y$$

$$v = e^{\frac{1}{3}y}$$

b) $v = e^{\frac{1}{3}y} = e^{\frac{y}{3}}$

$$u'e^{\frac{y}{3}} = \frac{y+1}{3}u^{-2}e^{-\frac{2y}{3}} \quad / e^{-\frac{y}{3}}\cdot u^2$$

$$u^2u' = \frac{y+1}{3}e^{-y}$$

$$u^2\frac{du}{dy} = \frac{1}{3}ye^{-y} + \frac{1}{3}e^{-y}$$

$$u^2du = \frac{1}{3}ye^{-y}dy + \frac{1}{3}e^{-y}dy \quad \ldots(1)$$

$$\int e^{-y}dy = \left|\begin{matrix} -y = t \\ dy = -dt \end{matrix}\right| = \int e^t(-dt) =$$

$$= -\int e^t dt = -e^t + C = -e^{-y} + C$$

Kako je $\int ye^{-y}dy = \left|\begin{matrix} u = y & dv = e^{-y}dy \\ du = dy & v = -e^{-y} \end{matrix}\right| = -ye^{-y} + \int e^{-y}dy = -ye^{-y} - e^{-y} + C$

To je kad izvadimo integral od (1):

$$\frac{1}{3}u^3 = -\frac{1}{3}ye^{-y} - \frac{1}{3}e^{-y} + C_1 - \frac{1}{3}e^{-y} \quad / \cdot 3$$

$$u^3 = -ye^{-y} - 2e^{-y} + C$$

$$u = \sqrt[3]{-ye^{-y} - 2e^{-y} + C}$$

$$x = uv$$

$$x = e^{\frac{y}{3}}\sqrt[3]{-ye^{-y} - 2e^{-y} + C}$$

opšte rješenje diferencijalne jednačine

$$x^3 = e^y(-ye^{-y} - 2e^{-y} + C)$$

$$x^3 = -y - 2 + Ce^y$$

(#) Riješiti diferencijalnu jednačinu $2x^3y' = 2x^2y - y^3$.

Rj.

$$2x^3y' = 2x^2y - y^3 \quad /:2$$

$$x^3y' - x^2y = \frac{-1}{2}y^3 \quad /:x^3$$

$$y' - \frac{1}{x}y = -\frac{1}{2x^3}y^3$$

Ovo je Bernulijeva diferencijalna jedn.
$(y' + p(x)y = g(x)y^n,\ n \in \mathbb{R},\ n \neq 0,\ n \neq 1)$

uvodimo smjenu $y = uv$

$y' = u'v + u \cdot v'$ gdje su $u$ i $v$ pomoćne f-je koje treba odrediti

$$u'v + uv' - \frac{1}{x}uv = -\frac{1}{2x^3}(uv)^3$$

$$u'v + \underbrace{u\left(v' - \frac{1}{x}v\right)}_{} = -\frac{1}{2x^3}u^3v^3$$

a) $v' - \frac{1}{x}v = 0$

$$v' = \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{v}{x}$$

$$\frac{dv}{v} = \frac{dx}{x}$$

$$\ln|v| = \ln|x|$$

$$v = x$$

b) $u'v + \underbrace{u\left(v' - \frac{v}{x}\right)}_{} = -\frac{1}{2x^3}u^3v^3$

ovo je jednako nuli za $v = x$

$$u'x = -\frac{1}{2x^3} \cdot u^3 \cdot x^3$$

$$u' = \frac{du}{dx}$$

$$\frac{du}{dx}x = -\frac{1}{2} \cdot u^3 \Rightarrow \frac{du}{u^3} = -\frac{1}{2} \cdot \frac{dx}{x} \quad //$$

$$\frac{u^{-2}}{-2} = -\frac{1}{2}\ln|x| + C_1 \quad /\cdot(-2)$$

$$\frac{1}{u^2} = \ln|x| + \ln C$$

$$u^2 = \frac{1}{\ln xC} \qquad u = \frac{1}{\sqrt{\ln xc}}$$

Rješenje diferencijalne jednačine je

$$y = \frac{x}{\sqrt{\ln xc}}$$

(#) Rješiti diferencijalnu jednačinu

$$\frac{dy}{dx}+\frac{y}{x}=y^2$$

Rj.

$$\frac{dy}{dx}+\frac{y}{x}=y^2$$

$y'+\frac{1}{x}y=y^2$ ovo je Bernulijeva diferencijalna jednačina

uvodimo smjenu $y=uv$

$y'=u'v+uv'$ (u i v su dvije pomoćne f-je)

$$u'v+uv'+\frac{1}{x}uv=u^2v^2$$

$$u'v+\underbrace{u\left(v'+\frac{1}{x}v\right)}_{=0}=u^2v^2$$

(a) $v'+\frac{1}{x}v=0$

$$\frac{dv}{dx}=\frac{-v}{x}$$

$$\frac{dv}{v}=-\frac{dx}{x}$$

$$\ln v=-\ln x$$

$$v=x^{-1}$$

$$v=\frac{1}{x}$$

(b) $u'v=u^2v^2$

$v=\frac{1}{x}$ (vidi (a))

$$u'\frac{1}{x}=u^2\frac{1}{x^2}\quad /x$$

$$u'=u^2\frac{1}{x}$$

$$\frac{du}{dx}=u^2\frac{1}{x}$$

$$\frac{du}{u^2}=\frac{1}{x}dx$$

$$\frac{u^{-1}}{-1}=\ln x+\ln c$$

$$-\frac{1}{u}=\ln xc$$

$$-u=\frac{1}{\ln xc}$$

$$u=\frac{-1}{\ln xc}$$

Opšte Rješenje date diferencijalne jednačine je $y=\frac{-1}{x\ln xc}$

(#) Rješiti diferencijalnu jednačinu

$$\frac{dy}{dx}+\frac{1}{3}y=e^{x}y^{4}$$

Rj:

$y'+\frac{1}{3}y=e^{x}y^{4}$ ovo je Bernulijeva diferencijalna jednačina
(uvodimo smjenu $y=uv$
$y'=u'v+uv'$ gdje su
u i v dvije pomoćne f-je)

$$u'v+uv'+\frac{1}{3}uv=e^{x}u^{4}v^{4}$$

$$u'v+u\underbrace{\left(v'+\frac{1}{3}v\right)}_{=0}=e^{x}u^{4}v^{4}$$

(a)

$$v'+\frac{1}{3}v=0$$

$$v'=-\frac{1}{3}v$$

$$\frac{dv}{dx}=-\frac{1}{3}v$$

$$\frac{dv}{v}=-\frac{1}{3}dx \quad /\int$$

$$\ln v=-\frac{1}{3}x$$

$$v=e^{-\frac{x}{3}}$$

(b)

$$u'v=e^{x}u^{4}v^{4}$$

$$u'\cdot e^{-\frac{x}{3}}=e^{x}\cdot u^{4}\cdot e^{-\frac{4}{3}x} \quad /\cdot e^{\frac{x}{3}}$$

$$\frac{du}{dx}=u^{4}$$

$$\frac{du}{u^{4}}=dx$$

$$\frac{u^{-3}}{-3}=x+C_1 \quad /\cdot(-3)$$

$$u^{-3}=-3x+C$$

$$u^{3}=\frac{1}{C-3x}$$

$$u=\frac{1}{\sqrt[3]{C-3x}}$$

Opšte
Rješenje date diferencijalne jednačine je

$$y=\frac{e^{-\frac{x}{3}}}{\sqrt[3]{C-3x}}$$

(#) Rješiti diferencijalnu jednačinu

$$x\frac{dy}{dx}+y=xy^3$$

Rj: $xy'+y=xy^3 \quad /:x$

$y'+\frac{1}{x}y=y^3$ ovo je Bernulijeva diferencijalna jednačina (uvodimo smjenu $y=uv$, $y'=u'v+uv'$)

$$u'v+uv'+\frac{1}{x}uv=u^3v^3$$

$$u'v+\underbrace{u\left(v'+\frac{v}{x}\right)}_{=0}=u^3v^3$$

(a) $v'+\frac{v}{x}=0$

$$\frac{dv}{dx}=-\frac{v}{x}$$

$$\frac{dv}{v}=-\frac{dx}{x}$$

$$\ln v=-\ln x$$

$$v=\frac{1}{x}$$

(b) $u'v=u^3v^3$

$$u'\frac{1}{x}=u^3\frac{1}{x^3} \quad /\cdot x$$

$$u'=\frac{u^3}{x^2}$$

$$\frac{du}{dx}=\frac{u^3}{x^2}$$

$$\frac{du}{u^3}=\frac{dx}{x^2}$$

$$\frac{u^{-2}}{-2}=\frac{x^{-1}}{-1}+C_1$$

$$\frac{1}{2u^2}=\frac{}{x}+C_2 \quad /\cdot 2$$

$$\frac{1}{u^2}=\frac{2}{x}+C$$

$$\frac{1}{u^2}=C+\frac{2}{x}$$

$$u^2=\frac{1}{C+\frac{2}{x}}$$

Opšte

Rješenje date diferencijalne jednačine je

$$y=\frac{1}{x}\cdot\frac{1}{\sqrt{C+\frac{2}{x}}}$$

# Lagranžova diferencijalna jednačina

su oblika $y = x f(y') + g(y')$ uvodimo smjene $y' = p$, $x = uv$

(1.) Riješiti diferencijalnu jednačinu $y + xy' = 4\sqrt{y'}$.

Rj. $y = x\cdot(-y') + 4\sqrt{y'}$ ovo je Lagr. dif. jedn.

uvodimo smjenu $y' = p$

$$y = -xp + 4\sqrt{p} \quad \Big/\frac{d}{dx}$$

$$y' = -p - xp' + 4\cdot\frac{1}{2\sqrt{p}}\cdot p' \ , \quad y' = p$$

$$2p = p'\left(-x + \frac{2}{\sqrt{p}}\right), \quad p' = \frac{dp}{dx}$$

$$\frac{1}{p'} = \frac{dx}{dp} = x' \qquad \frac{1}{p'} = \frac{-x + \frac{2}{\sqrt{p}}}{2p}$$

$$x' = -\frac{x}{2p} + \frac{1}{p\sqrt{p}}$$

$$x' + \frac{x}{2p} = \frac{1}{p\sqrt{p}}$$ ovo je linear. dif. jedn.

uvodimo smjenu $x = uv$,
$x' = u'v + uv'$

$$u'v + uv' + \frac{uv}{2p} = \frac{1}{p\sqrt{p}}$$

$$u'v + u\underbrace{\left(v' + \frac{v}{2p}\right)}_{=0} = \frac{1}{p\sqrt{p}}$$

a) $v' + \frac{v}{2p} = 0$, $\frac{dv}{dp} = -\frac{v}{2p}$

$$\frac{dv}{v} = -\frac{1}{2}\,\frac{dp}{p} \quad //$$

$$\ln|v| = -\frac{1}{2}\ln|p|$$

$$v = p^{-\frac{1}{2}} = \frac{1}{\sqrt{p}}$$

b) $u'\cdot\frac{1}{\sqrt{p}} = \frac{1}{p\sqrt{p}} \ \Big/\cdot\sqrt{p} \ \Rightarrow u' = \frac{1}{p}$

$$u = \ln|p| + c$$

$$x = u\cdot v = \frac{\ln|p| + c}{\sqrt{p}} \quad (*)$$

$$y = -xp + 4\sqrt{p} = -p\,\frac{c + \ln|p|}{\sqrt{p}} + 4\sqrt{p}$$

$$y = \sqrt{p}\,(4 - c - \ln|p|) \quad (**)$$

(*) i (**) je rješenje dif. jedn. u parametarskom obliku

(2.) Riješiti diferencijalnu jednačinu

$y'(2x - y) = y$. Rj.

$$\left.\begin{aligned} x &= \frac{2}{3}p + \frac{c}{p^2} \\ y &= 2xp - p^2 \end{aligned}\right\}$$ opšte rješ. dif. jedn. u parametarskom obliku

(3.) Naći rješenje diferencijalne jednačine $y = xy' - 2 - y'$ koje prolazi kroz tačku $A(2,5)$.

Rj. $y = xc - 2 - c$ opšte rješenje $\qquad$ $y = 7x - 9$ partikularno rješenje

(#) Riješiti diferencijalnu jednačinu $2y+y'(2x+y')=0$.

Rj.

$y=-xy'-\frac{1}{2}(y')^2$ ovo je Lagranžova diferenc. jedn. $(y=xf(y')+g(y'))$

uvodimo smjene $y'=p$, $x=uv$

$y'=p$

$$y=-xp-\frac{1}{2}p^2 \quad \Big/ \frac{d}{dx}$$

$$y'=-p-xp'-\frac{1}{2}\cdot 2pp'$$

$$p=-p-xp'-pp'$$

$$2p=(-x-p)p' \quad /:p'$$

$$\frac{2p}{p'}=-x-p, \quad p'=\frac{dp}{dx}$$

$$\frac{1}{p'}=\frac{dx}{dp}=x'$$

$$x'=-\frac{1}{2p}x-\frac{1}{2}$$

$$x'+\frac{1}{2p}x=-\frac{1}{2}$$

ovo je linearna dif. jedn. $(y'+f(x)y=g(x))$

uvodimo smjenu

$x=uv, \quad x'=u'v+uv'$

$$u'v+uv'+\frac{1}{2p}uv=-\frac{1}{2}$$

$$u'v+u\underbrace{\left(v'+\frac{1}{2p}v\right)}_{=0}=-\frac{1}{2}$$

$$v'+\frac{1}{2p}v=0$$

$$\frac{dv}{dp}=-\frac{1}{2p}v$$

$$\frac{dv}{v}=-\frac{1}{2}\cdot\frac{dp}{p} \quad \Big/\int$$

$$\ln|v|=-\frac{1}{2}\ln|p|$$

$$v=p^{-\frac{1}{2}}=\frac{1}{\sqrt{p}}$$

$$u'v=-\frac{1}{2}$$

$$u'\cdot\frac{1}{\sqrt{p}}=-\frac{1}{2}$$

$$\frac{du}{dp}=-\frac{1}{2}\sqrt{p}$$

$$du=-\frac{1}{2}p^{\frac{1}{2}}dp \quad \Big/\int$$

$$u=-\frac{1}{2}\cdot\frac{p^{\frac{3}{2}}}{\frac{3}{2}}+c$$

$$u=-\frac{1}{2}\cdot\frac{2}{3}\sqrt{p^3}+c$$

$$u=-\frac{1}{3}p\sqrt{p}+c$$

$$x=uv=\left(-\frac{1}{3}p\sqrt{p}+c\right)\cdot\frac{1}{\sqrt{p}}$$

$$x=-\frac{p}{3}+\frac{c}{\sqrt{p}}$$

$$y=-xp-\frac{1}{2}p^2$$

$$y=\left(\frac{p}{3}-\frac{c}{\sqrt{p}}\right)p-\frac{1}{2}p^2$$

$$y=-\frac{1}{6}p^2-\frac{p}{\sqrt{p}}c$$

$$\left.\begin{aligned} x&=-\frac{p}{3}+\frac{c}{\sqrt{p}} \\ y&=-\frac{p^2}{6}-\frac{p}{\sqrt{p}}c \end{aligned}\right\} \text{opće rješenje diferencijalne jednačine}$$

(#) Riješiti diferencijalnu jednačinu $y'+\frac{1}{y'}=\frac{y}{x}$.

Rj.

$$y'+\frac{1}{y'}=\frac{y}{x} \quad /\cdot x$$

$$y=xy'+\frac{x}{y'}$$ uvodimo smjenu $y'=p$

$$y=xp+\frac{x}{p} \quad /\frac{d}{dx}$$

$$y'=p+xp'+\frac{p-xp'}{p^2}$$ (kako je $y'=p$ ~~imamo~~)

$$p=p+xp'+\frac{1}{p^2}(p-xp')$$

$$xp'+\frac{1}{p}-\frac{xp'}{p^2}=0$$

$$\left(x-\frac{x}{p^2}\right)p'=-\frac{1}{p} \quad /\cdot p$$

$$\left(px-\frac{x}{p}\right)p'=-1 \quad /\cdot\frac{1}{p'}$$

$$-\frac{1}{p'}=px-\frac{1}{p}x$$

$$-\frac{1}{p'}=\left(p-\frac{1}{p}\right)x$$

znamo da je $\frac{1}{p'}=\frac{1}{\frac{dp}{dx}}=\frac{dx}{dp}=x'$

pa imamo

$$-x'=\left(p-\frac{1}{p}\right)x \quad /\cdot(-1)$$

$$x'=\left(\frac{1}{p}-p\right)x$$ ovo je diferencijalna jednačina sa razdvojenim promjenjivim

$$y=xy'+\frac{x}{y'}$$

$$y=x\left(y'+\frac{1}{y'}\right)$$

ovo je Lagranžova diferencijalna jednačina

$$x'-\left(\frac{1}{p}-p\right)x$$

$$\frac{dx}{dp}=\left(\frac{1}{p}-p\right)x$$

$$\frac{dx}{x}=\left(\frac{1}{p}-p\right)dp \quad //$$

$$\int\frac{dx}{x}=\int\left(\frac{1}{p}-p\right)dp.$$

$$\ln|x|=\ln|p|-\frac{p^2}{2}+C_1$$

$$\ln|x|=\ln|p|+\ln e^{-\frac{p^2}{2}}+\ln C$$

$$x=pCe^{-\frac{p^2}{2}}$$

$$y=xp+\frac{x}{p}=Cpe^{-\frac{p^2}{2}}\cdot p+\frac{pCe^{-\frac{p^2}{2}}}{p}$$

$$y=Cp^2e^{-\frac{p^2}{2}}+C\cdot e^{-\frac{p^2}{2}}$$

$$y=Ce^{-\frac{p^2}{2}}(p^2+1)$$

$$\left.\begin{aligned}x&=pCe^{-\frac{p^2}{2}}\\ y&=Ce^{-\frac{p^2}{2}}(p^2+1)\end{aligned}\right\}$$ opšte rješenje

# Clairautova diferencijalna jednačina

je oblika $y = xy' + f(y')$

ove diferencijalne jednačine rješavamo na isti način kao što smo rješavali Lagranžovu dif. jedn.

uvodimo smjenu $y' = p$ , $x = uv$
$dy = p\,dx$

① Riješiti diferencijalnu jednačinu $xy' + \sin y' - y = 0$

Rj. $y = xy' + \sin y'$

$$y' = p \Rightarrow \frac{dy}{dx} = p \Rightarrow dy = p\,dx$$

$$y = xp + \sin p \quad / d$$

$$dy = p\,dx + x\,dp + \cos p\,dp$$

$$\underline{\underline{p\,dx}} = \underline{\underline{p\,dx}} + x\,dp + \cos p\,dp$$

$$x\,dp + \cos p\,dp = 0$$

$$dp\,(x + \cos p) = 0$$

$$dp = 0 \Rightarrow p = c$$

$y = cx + \sin c$ opšte rješenje dif. jedn.

② Riješiti diferencijalnu jednačinu $y - xy' - \frac{y'^2}{2} = 0$.

Rj. $y = cx + \frac{c^2}{2}$ opšte rješenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu $2y-2xy'=a(\sqrt{1+(y')^2}-y')$.

Rj: Lagranžova diferencijalna jednačina je oblika $y=xf(y')+g(y')$

$$2y-2xy'=a(\sqrt{1+(y')^2}-y')$$

$$2y=2xy'+a(\sqrt{1+(y')^2}-y') \quad /:2$$

$$y=xy'+\frac{a}{2}(\sqrt{1+(y')^2}-y')$$

Ovo je Klerova diferencijalna jednačina

Uvodimo smjenu $y'=p$

$$y=xp+\frac{a}{2}(\sqrt{1+p^2}-p) \quad \Big/ \frac{d}{dx}$$

$$y'=p+xp'+\frac{a}{2}\left(\frac{\cancel{2}pp'}{\cancel{2}\sqrt{1+p^2}}-p'\right)$$

$y'=p$

$$p=p+xp'+\frac{a}{2}p'\left(\frac{p}{\sqrt{1+p^2}}-1\right)$$

$$-xp'=\frac{a}{2}p'\left(\frac{p}{\sqrt{1+p^2}}-1\right)$$

$$\left[x+\frac{a}{2}\left(\frac{p}{\sqrt{1+p^2}}-1\right)\right]p'=0$$

a) Ako je $p'=0$ imamo da je $p=c$ tj. $y'=c$ pa iz $y=xy'+\frac{a}{2}(\sqrt{1+(y')^2}-y')$

$$\Rightarrow y=xc+\frac{a}{2}(\sqrt{1+c^2}-c)$$

$y=xC_1+\frac{a}{2}C_2$ opšte rješenje diferencijalne jednačine

---

b) Ako je

$$x+\frac{a}{2}\left(\frac{p}{\sqrt{1+p^2}}-1\right)=0$$

$$\frac{p}{\sqrt{1+p^2}}-1=-\frac{2}{a}x$$

$$\frac{p}{\sqrt{1+p^2}}=1-\frac{2x}{a}$$

$$p^2=\left(1-\frac{2x}{a}\right)^2(1+p^2)$$

$$p^2-\left(1-\frac{2x}{a}\right)^2p^2=\left(1-\frac{2x}{a}\right)^2$$

$$p^2=\frac{\left(1-\frac{2x}{a}\right)^2}{1-\left(1-\frac{2x}{a}\right)^2}$$

$$p=\frac{1-\frac{2x}{a}}{\sqrt{1-\left(1-\frac{2x}{a}\right)^2}}$$

$$y=xy'+\frac{a}{2}\left(\sqrt{1+(y')^2}-y'\right)\Rightarrow$$

$$\Rightarrow y=\frac{x-\frac{2}{a}x^2}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}+\frac{a}{2}\left(\sqrt{1+\frac{\left(1-\frac{2}{a}x\right)^2}{1-\left(1-\frac{2x}{a}\right)^2}}-\frac{1-\frac{2x}{a}}{\sqrt{1-\left(1-\frac{2x}{a}\right)^2}}\right)$$

kako se ovo rješenje ne može dobiti iz opšteg rješenja ovo je singularno rješenje

zadnji izraz se može pojednostaviti

$$Y=\frac{x-\frac{2}{a}x^2}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}+\frac{a}{2}\left(\sqrt{\frac{1-\cancel{\left(1-\frac{2}{a}x\right)^2}+\cancel{\left(1-\frac{2}{a}x\right)^2}}{1-\left(1-\frac{2}{a}x\right)^2}}-\frac{1-\frac{2}{a}x}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}\right)$$

$$Y=\frac{x-\frac{2}{a}x^2}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}+\frac{\frac{a}{2}}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}-\frac{\frac{a}{2}-x}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}.$$

$$Y=\frac{2x-\frac{2}{a}x^2}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}$$

singularno
rjesenje
dif. jedn.

(#) Riješiti diferencijalnu jednačinu $y-xy'-\frac{1}{2}y'^2=0$.

Rj.

$$y-xy'-\frac{1}{2}y'^2=0$$

$$y=xy'+\frac{1}{2}y'^2$$

Jednačine oblika $y=xy'+f(y')$ se nazivaju Clairaut-ove dif. jed

uvodimo smjenu $y'=p$

$$dy=p\,dx$$

$$y=xp+\frac{1}{2}p^2 \quad /d$$

$$\underbrace{dy}_{=p\,dx}=\underbrace{p\,dx}+x\,dp+p\,dp$$

$$(x+p)\,dp=0$$

(a) $dp=0$

$$p=C$$

$$y=xC+\frac{1}{2}C^2$$

je opšte rješenje diferencijalne jednačine

(b) $x+p=0$

$$p=-x$$

$$y=xp+\frac{1}{2}p^2$$

$$y=-x^2+\frac{1}{2}x^2$$

$$y=-\frac{1}{2}x^2$$

singularno rješenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu $y'^2 - xy' + y = 0$.

Rj:

$$y'^2 - xy' + y = 0$$

$$y = xy' - y'^2$$

Jednačina oblika $y = xy' + f(y')$ se naziva Clairautova dif.jedn.

uvodimo smjenu $y' = p \Rightarrow dy = p\,dx$

$$y' = \frac{dy}{dx}$$

$$y = xp - p^2 \quad /d$$

$$dy = p\,dx + x\,dp - 2p\,dp$$

$$\underline{p\,dx} = \underline{p\,dx} + x\,dp - 2p\,dp$$

$$(x - 2p)\,dp = 0$$

(a) $dp = 0$

$p = c \Rightarrow y = xc - c^2$

je opšte rješenje
diferencijalne
jednačine

(b) $x - 2p = 0$

$$2p = x$$

$$p = \frac{x}{2}$$

$$y = xp - p^2$$

$$y = \frac{1}{2}x^2 - \frac{1}{4}x^2$$

$$y = \frac{1}{4}x^2$$

singularno
rješenje
diferencij. jedn.

(#) Riješiti diferencijalnu jednačinu $(y-y'x)^2=1+y'^2$.

Rj: $y-y'x=\pm\sqrt{1+y'^2}$

$$y=y'x\pm\sqrt{1+y'^2}$$

Jednačina oblika $y=xy'+f(y')$ se naziva Clairat-ova dif. jedn. i ove diferencijalne jednačine rješavamo na potpuno isti način kao što smo rješavali Lagrange-ove difer. jednač. uvodimo smjenu $y'=p$. ($dy=p\,dx$, $x=uv$).

$y=y'x\pm\sqrt{1+y'^2}$ ovo je Clairaut-ova dif. jedn.

$y'=p$, $y'=\frac{dy}{dx}$ ⇒ $dy=p\,dx$

$y=px\pm\sqrt{1+p^2}$ /d

$$\underset{=p\,dx}{\underline{dy}}=\underline{p\,dx}+x\,dp\pm\frac{2p}{2\sqrt{1+p^2}}dp$$

$$\left(x\pm\frac{p}{\sqrt{1+p^2}}\right)dp=0$$

(a) $dp=0$

$p=c$

$(y-cx)^2=1+c^2$

opšte rješenje diferencijalne jedn.

(b) $x\pm\frac{p}{\sqrt{1+p^2}}=0$

$\pm\frac{p}{\sqrt{1+p^2}}=-x$ $/^2$

$$\frac{p^2}{1+p^2}=x^2$$

$$p^2=\underbrace{x^2(1+p^2)}_{x^2+x^2p^2}$$

$$(1-x^2)p^2=x^2$$

$$p^2=\frac{x^2}{1-x^2}$$

$$p=\frac{x}{\sqrt{1-x^2}}$$

$y=\frac{x^2}{\sqrt{1-x^2}}\pm\sqrt{1+\frac{x^2}{1-x^2}}=\frac{x^2\pm1}{\sqrt{1-x^2}}$ singularno rješenje diferenc. jednačine

(#) Riješiti diferencijalnu jednačinu $y = y'x + \sqrt{4+y'^2}$.

Rj. Jednačina oblika $y = xy' + f(y')$ se naziva Clairaut-ova diferencijalna jednačina i ove doferencijalne jednačine rješavamo na potpuno isti način kao što smo rješavali Lagrange-ove diferencijalne jednačine - uvodimo smjenu

$y' = p \,, \quad \left| y' = \frac{dy}{dx} \right|$

$dy = p\,dx$

$x = uv$

---

$y = y'x + \sqrt{4+y'^2}$ Clair. dif. jedn.

$y' = p \Rightarrow dy = p\,dx$

$y = px + \sqrt{4+p^2} \quad / d$

$dy = p\,dx + x\,dp + \frac{2p\,dp}{2\sqrt{4+p^2}}$

$p\,dx = p\,dx + x\,dp + \frac{p\,dp}{\sqrt{4+p^2}}$

$\left(x + \frac{p}{\sqrt{4+p^2}}\right) dp = 0$

(a) $dp = 0 \Rightarrow p = c$

$y = cx + \sqrt{4+c^2}$

opšte rješenje diferenc. jednačine

(b) $x + \frac{p}{\sqrt{4+p^2}} = 0$

$\frac{p}{\sqrt{4+p^2}} = -x \quad /^2$

$\frac{p^2}{4+p^2} = x^2$

$p^2 = x^2(4+p^2)$

$p^2 = x^2p^2 + 4x^2$

$(1-x^2)p^2 = 4x^2$

$p^2 = \frac{4x^2}{1-x^2}$

$p = \frac{2x}{\sqrt{1-x^2}}$

$y = px + \sqrt{4+p^2} = \frac{2x^2}{\sqrt{1-x^2}} + \sqrt{4 + \frac{4x^2}{1-x^2}} = \frac{2x^2}{\sqrt{1-x^2}} + \sqrt{\frac{4-4x^2+4x^2}{1-x^2}}$

$y = \frac{2x^2+2}{\sqrt{1-x^2}}$ singularno rješenje diferencijalne jednačine (rješenje koje se ne može dobiti iz opšteg rješenj